The art of modeling water waves

Habilitation à Diriger des Recherches

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Why "modeling"?

Replace "complicated' set of equations with "simple" set of equations.

1. To enlighten the basic mechanisms of a phenomenon
   - Wavebreaking: $\partial_t u + u\partial_x u = 0$ (Hopf)
   - Solitary waves: $\partial_t u + u\partial_x u + \partial_x^3 u = 0$ (KdV)
   - Non-smooth solitary waves (or wave breaking and solitary waves):

$$\partial_t u + \sqrt{\frac{\tanh(|D|)}{|D|}}\partial_x\zeta + \zeta\partial_x\zeta = 0 \qquad \text{(Whitham)}$$

2. To produce approximate solutions (e.g. numerical)
   - $\mathcal{O}(\mu)$: $\partial_t\zeta + \sqrt{gd}\big(\partial_x\zeta + \frac{3}{2d}\zeta\partial_x\zeta\big) = 0$ (Hopf)
   - $\mathcal{O}(\mu^2 + \mu\epsilon)$: $\partial_t\zeta + \sqrt{gd}\big(\partial_x\zeta + \frac{3}{2d}\zeta\partial_x\zeta + d^2\partial_x^3\zeta\big) = 0$ (KdV)
   - $\mathcal{O}(\mu\epsilon)$: $\partial_t\zeta + \sqrt{gd}\Big(\sqrt{\frac{\tanh(d|D|)}{d|D|}}\partial_x\zeta + \frac{3}{2d}\zeta\partial_x\zeta\Big) = 0$ (Whitham)
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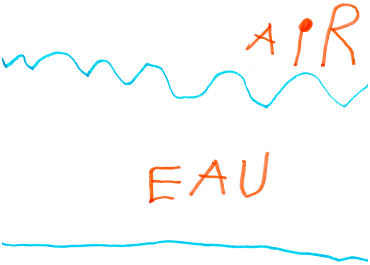

Figure: Water waves, by Anouk and Lucie Duchêne

[Feynman] "[water waves] that are easily seen by everyone and which are usually used as an example of waves in elementary courses [...] are the worst possible example [...]; they have all the complications that waves can have."

Standard models include: Saint-Venant, Boussinesq, Serre–Green–Naghdi, Matsuno, Korteweg–de Vries, Benjamin–Bona–Mahony, Camassa–Holm, Kawahara, Whitham, Kadomtsev–Petviashvili, Dysthe, Benney–Roskes, NLS...

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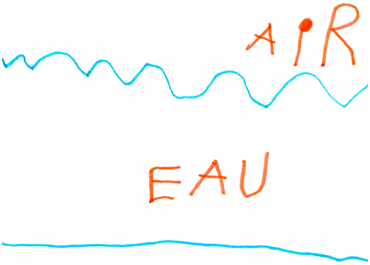

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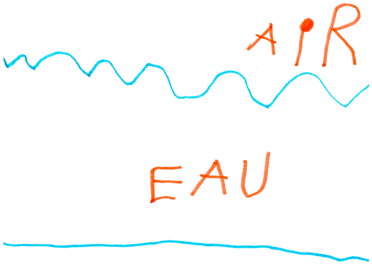

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Why “the art”?

There will be traps. Avoiding them will have a cost.
We will make choices, with benefits and downsides.

A useful tool: theorems.

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Table of contents

All you need to know about the water waves system (today)

Warning: the following applies only to inviscid, incompressible, homogeneous, irrotational flows. Serving suggestion.

Zakharov/Craig-Sulem formulation [Zakharov '68, Craig&Sulem '93]

$$\begin{cases} \partial_t \zeta - \frac{\delta \mathscr{H}}{\delta \psi} = 0, \\ \partial_t \psi + \frac{\delta \mathscr{H}}{\delta \zeta} = 0, \end{cases} \qquad \text{(WW)}$$

with

$$\mathscr{H}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi\, \mathcal{G}^\mu[\epsilon\zeta]\psi \,\mathrm{d}\mathbf{x}$$

where the **Dirichlet-to-Neumann operator** $\mathcal{G}^\mu[\epsilon\zeta]\psi$ is defined by

$$\mathcal{G}^\mu[\epsilon\zeta]\psi = \left(\tfrac{1}{\mu}\partial_z \Phi - \epsilon \nabla\zeta \cdot \nabla_{\mathbf{x}}\Phi\right)\big|_{z=\epsilon\zeta}$$

with Φ solution to

$$\begin{cases} \mu\Delta_{\mathbf{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \{(\mathbf{x},z),\ -1 < z < \epsilon\zeta(t,\mathbf{x})\}, \\ \Phi = \psi & \text{on } \{(\mathbf{x},z),\ z = \epsilon\zeta(t,\mathbf{x})\}, \\ \partial_z\Phi = 0 & \text{on } \{(\mathbf{x},z),\ z = -1\}. \end{cases}$$

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$$\boxed{\text{Water waves} \quad = \quad (\ \text{Hyperbolic}\ ) \ \times \ (\ \text{Elliptic}\ ).}$$

Our journey starts with ripples

We set $\epsilon = 0$

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$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}} |D| \tanh(\sqrt{\mu}|D|)\psi = 0, \\ \partial_t \psi + \zeta = 0, \end{cases} \qquad \text{(Airy)}$$

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Lessons from the modal analysis

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Dispersion relation (for plane waves $\propto e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$)

$$c^2 \stackrel{\text{def}}{=} \frac{\omega^2}{|\mathbf{k}|^2} = \frac{\tanh(\sqrt{\mu}|\mathbf{k}|)}{\sqrt{\mu}|\mathbf{k}|}.$$

Some approximations (valid when $\sqrt{\mu}|\mathbf{k}| \ll 1$)

1. $c^2 = 1$ ✓ Non-dispersive. Relative error less than 10% for $\sqrt{\mu}|\mathbf{k}| < 0.055$.
2. $c^2 = 1 - \frac{\mu}{3}|\mathbf{k}|^2$ ✗ Unstable modes for $\sqrt{\mu}|\mathbf{k}| > \sqrt{3}$.
3. $c^2 = \frac{1}{1+\frac{\mu}{3}|\mathbf{k}|^2}$ ✓ Relative error less than 10% for $\sqrt{\mu}|\mathbf{k}| < 1.914$.
4. $c^2 = 1 - \frac{\mu}{3}|\mathbf{k}|^2 + \frac{2\mu^2}{15}|\mathbf{k}|^4 + \dots$ ✗ Series do not converge for $\sqrt{\mu}|\mathbf{k}| > \frac{\pi}{2}$.
5. $c^2 = \frac{1}{1+\frac{\mu}{3}|\mathbf{k}|^2-\frac{\mu^2}{45}|\mathbf{k}|^4+\dots}$ ✗ Series do not converge for $\sqrt{\mu}|\mathbf{k}| > \pi$.
6. $c^2 = \cfrac{1}{1+\cfrac{\mu|\mathbf{k}|^2}{3+\cfrac{\mu|\mathbf{k}|^2}{5+\cfrac{\mu|\mathbf{k}|^2}{\dots}}}}$ ✓ Padé approximant. Uniformly convergent.

Switching back on the nonlinearity

Recall the **Dirichlet-to-Neumann operator** $\mathcal{G}^\mu[\epsilon\zeta]\psi$ is defined by

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An equivalent formulation is

$$\mathcal{G}^\mu[\epsilon\zeta]\psi = -\nabla\cdot\left(\int_{-1}^{\epsilon\zeta(t,\cdot)}\nabla_{\mathbf{x}}\Phi(\cdot,z)\,\mathrm{d}z\right)$$

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$$\Phi + \mu\ell[\epsilon\zeta]\Phi = \psi, \qquad \ell[\epsilon\zeta]\Phi(\cdot,z) \stackrel{\text{def}}{=} -\int_z^{\epsilon\zeta}\int_{-1}^{z'}\Delta_{\mathbf{x}}\Phi(\cdot,z'')\,\mathrm{d}z''\,\mathrm{d}z'.$$

We infer approximate formula at any order $\mathcal{O}(\mu^N)$:

$$\Phi = \psi + \mathcal{O}(\mu), \qquad \Phi = \psi - \mu\ell[\epsilon\zeta]\psi + \mathcal{O}(\mu^2), \qquad \dots$$

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This yields approximations to the Dirichlet-to-Neumann operator:

✓ $$\mathcal{G}^{\mu}[\epsilon\zeta]\psi = -\nabla\cdot((1+\epsilon\zeta)\nabla\psi) + \mathcal{O}(\mu),$$

✗ $$\mathcal{G}^{\mu}[\epsilon\zeta]\psi = -\nabla\cdot(h\nabla\psi) + \mu\nabla\cdot\big(h\mathcal{T}[h]\nabla\psi\big) + \mathcal{O}(\mu^2),$$

✓ $$\mathcal{G}^{\mu}[\epsilon\zeta]\psi = -\nabla\cdot\Big(h\big(\mathrm{Id} + \mu\mathcal{T}[h]\big)^{-1}\nabla\psi\Big) + \mathcal{O}(\mu^2),$$

with $h = 1+\epsilon\zeta$ and $\mathcal{T}[h]\mathbf{u} = \frac{-1}{3h}\nabla(h^3\nabla\cdot\mathbf{u})$.

Approximations to the Dirichlet-to-Neumann operator

[Lannes]

For any sufficiently regular ζ such that

$$\forall \mathbf{x} \in \mathbb{R}^d, \qquad h(\mathbf{x}) \stackrel{\text{def}}{=} 1 + \epsilon\zeta(\mathbf{x}) \geq h_\star > 0,$$

one has for any $k \in \mathbb{N}$, $\epsilon \geq 0$ and $\mu \in (0,1]$,

✓ $$\left|\mathcal{G}^\mu[\epsilon\zeta]\psi + \nabla\cdot((1+\epsilon\zeta)\nabla\psi)\right|_{H^k} \leq C_{k+4}\,\mu\,\left|\nabla\psi\right|_{H^{k+3}},$$

✗ $$\left|\mathcal{G}^\mu[\epsilon\zeta]\psi + \nabla\cdot(h\nabla\psi) - \mu\nabla\cdot\left(h\mathcal{T}[h]\nabla\psi\right)\right|_{H^k} \leq C_{k+6}\,\mu^2\,\left|\nabla\psi\right|_{H^{k+5}},$$

✓ $$\left|\mathcal{G}^\mu[\epsilon\zeta]\psi + \nabla\cdot\Big(h\big(\operatorname{Id}+\mu\mathcal{T}[h]\big)^{-1}\nabla\psi\Big)\right|_{H^k} \leq C_{k+6}\,\mu^2\,\left|\nabla\psi\right|_{H^{k+5}},$$

with $C_k = C(k, h_\star^{-1}, \left|\epsilon\zeta\right|_{H^n})$ and $\mathcal{T}[h]\mathbf{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\mathbf{u})$.

Plugging these approximations in the water waves equations yields...

Historical shallow-water models

The Saint-Venant system

$$\begin{cases} \partial_t \zeta + \nabla \cdot \big((1+\epsilon\zeta)\mathbf{u}\big) = 0, \\ \partial_t \mathbf{u} + \nabla\zeta + \epsilon(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0}, \end{cases} \tag{SV}$$

with $\mathbf{u} = \nabla\psi$ (or $\mathbf{u} = \overline{\mathbf{u}} \stackrel{\text{def}}{=} \frac{1}{1+\epsilon\zeta}\int_{-1}^{\epsilon\zeta} \nabla_{\mathbf{x}}\Phi(\cdot, z)\,\mathrm{d}z$).

The Green–Naghdi system

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0, \\ \big(\mathrm{Id} + \mu\mathcal{T}[h]\big)\partial_t \mathbf{u} + \nabla\zeta + \epsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\epsilon\mathcal{Q}[h,\mathbf{u}] = \mathbf{0}, \end{cases} \tag{GN}$$

where $h \stackrel{\text{def}}{=} 1 + \epsilon\zeta$, $\mathcal{Q}[h,\mathbf{u}] \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2\big)\Big)$, and $\mathcal{T}[h]\mathbf{u} \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla(h^3\nabla\cdot\mathbf{u})$ with $\mathbf{u} = \big(\mathrm{Id} + \mu\mathcal{T}[h]\big)^{-1}\nabla\psi$ (or $\mathbf{u} = \overline{\mathbf{u}}$).

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with $\mathbf{u} = \nabla\psi$ (or $\mathbf{u} = \overline{\mathbf{u}} \stackrel{\rm def}{=} \frac{1}{1+\epsilon\zeta}\int_{-1}^{\epsilon\zeta} \nabla_{\mathbf{x}}\Phi(\cdot,z)\,\mathrm{d}z$).
A special case of *compressible* Euler equations. Finite-time singularity formation.
Used when the problem features dry zones, discontinuities (dam-break), *etc.*

The Green–Naghdi system

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Explicit family of solitary waves. Globally well-posed?
A *lot* of activity around (GN) recently.

Fully rigorous justification

The full justification of a model typically stems from the combination of

1. Consistency

   Regular solutions to the water waves system satisfy approximately the model
2. Well-posedness

   Existence and control of solutions on a relevant time interval
3. Stability

   Control of the difference between exact and approximate solutions of the model

$\leadsto$ Control of $\mathfrak{e}$, the difference between the solution to the water waves system and the corresponding solution to the model.

The Saint-Venant system is a quasilinear hyperbolic symmetrizable system.

$\leadsto$[Friedrichs,Garding,Kato '50s] WP and Stability in $H^s(\mathbb{R}^d)^{1+d}$, $s > 1 + d/2$

$$\big|\mathfrak{e}_{\rm SV}\big|_{H^k} \lesssim \mu\, t, \qquad t \lesssim 1/\epsilon.$$

Fully rigorous justification

The full justification of a model typically stems from the combination of

1. Consistency

   Regular solutions to the water waves system satisfy approximately the model
2. Well-posedness

   Existence and control of solutions on a relevant time interval
3. Stability

   Control of the difference between exact and approximate solutions of the model

$\rightsquigarrow$ Control of $\mathfrak{e}$, the difference between the solution to the water waves system and the corresponding solution to the model.

The Saint-Venant system is a quasilinear hyperbolic symmetrizable system.
$\rightsquigarrow$[Friedrichs,Garding,Kato '50s] WP and Stability in $H^s(\mathbb{R}^d)^{1+d}$, $s > 1 + d/2$

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The Green–Naghdi system is a "quasilinear hyperbolic symmetrizable system".
$\rightsquigarrow$ [Li '06, Fujiwara&Iguchi '15] WP and Stability in $H^s(\mathbb{R}^d) \times X^s$, $s > 1 + d/2$

$$X^s \stackrel{\rm def}{=} \{\mathbf{u} \ : \ \big|\mathbf{u}\big|_{X^s}^2 \stackrel{\rm def}{=} \big|\mathbf{u}\big|_{H^s}^2 + \mu\big|\nabla\cdot\mathbf{u}\big|_{H^s}^2 < \infty\}.$$

$$\big|\mathfrak{e}_{\rm GN}\big|_{H^k\times X^k} \lesssim \mu^2\, t, \qquad t \lesssim 1/\epsilon.$$

Hyperbolic reformulation

Recall **The Green–Naghdi system**

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0, \\ \big(\operatorname{Id} + \mu \mathcal{T}[h]\big)\partial_t \mathbf{u} + \nabla\zeta + \epsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\epsilon\mathcal{Q}[h,\mathbf{u}] = \mathbf{0}, \end{cases} \qquad \text{(GN)}$$

where $h \stackrel{\text{def}}{=} 1 + \epsilon\zeta$, $\mathcal{T}[h]\mathbf{u} = \frac{-1}{3h}\nabla(h^3\nabla\cdot\mathbf{u})$ and
$\mathcal{Q}[h,\mathbf{u}] \stackrel{\text{def}}{=} \frac{-1}{3h}\nabla\Big(h^3\big((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2\big)\Big)$.

In numerical simulations, we need to solve at each timestep (for $\mathbf{u}$)

$$\big(\operatorname{Id} + \mu\mathcal{T}[h]\big)\mathbf{u} = \mathbf{v}.$$

The Green–Naghdi system can be written as

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3 evolution equations + constraint. $\rightsquigarrow$ *relaxation methods*.

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[Favrie&Gavrilyuk '17] proposed

$$\begin{cases} \partial_t \zeta + \nabla \cdot ((1+\epsilon\zeta)\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla \zeta + \epsilon(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{\lambda\mu}{3h}\nabla\Big(\frac{1+\epsilon\eta}{1+\epsilon\zeta}(\eta - \zeta)\Big) = \mathbf{0}, \\ \partial_t w + \epsilon \mathbf{u} \cdot \nabla w = -\frac{\lambda}{h^2}(\eta - \zeta), \\ \partial_t \eta + \epsilon \mathbf{u} \cdot \nabla \eta = w. \end{cases} \tag{FG}$$

Quasilinear system of balance laws, *singular limit* $\lambda \gg 1$ and $\mu \ll 1$.

[VD '19]: rigorous justification for well-prepared initial data and $\lambda \gtrsim \mu^{-1}$.

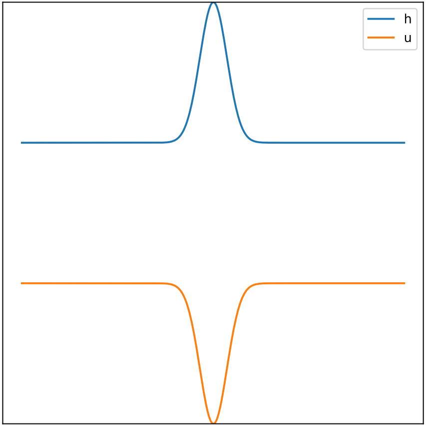
$$\big|\mathfrak{e}_{\rm FG}\big|_{H^k \times X^k} \lesssim (\mu^2 + \mu\lambda^{-1})\, t, \qquad t \lesssim 1/\epsilon.$$

Toy models

Order 1

$$\begin{cases} \partial_t h = 0, \\ \partial_t u + \frac{1}{\epsilon} h \partial_x u = 0 \end{cases}$$

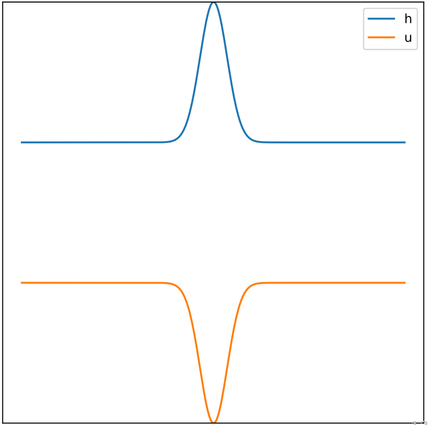
▸ Animation



Order 0

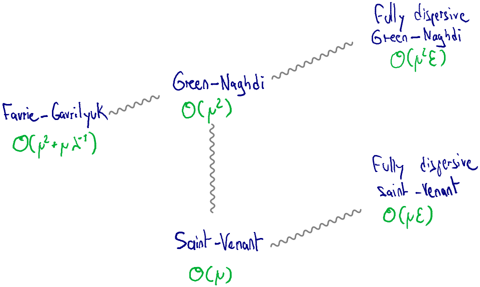
$$\begin{cases} \partial_t h = 0, \\ \partial_t u + \frac{i}{\epsilon} h u = 0 \end{cases}$$

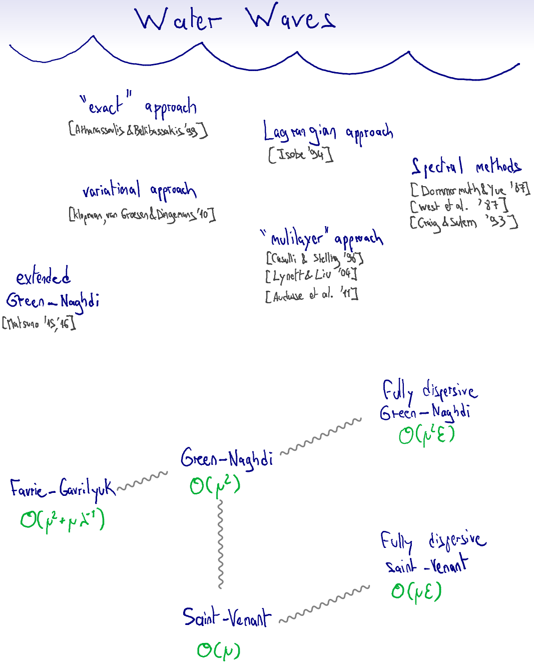
▸ Animation



Water Waves

High order models





Water Waves
"exact" approach
[Athanassoulis & Belibassakis '99]
Lagrangian approach
[Isobe '94]
Spectral methods
[Dommermuth & Yue '87]
[West et al. '87]
[Craig & Sulem '93]
variational approach
[Klopman, van Groesen & Dingemans '10]
"multilayer" approach
[Casulli & Stelling '98]
[Lynett & Liu '04]
[Audusse et al. '11]
extended Green–Naghdi
[Matsuno '15, '16]
Fully dispersive Green–Naghdi
O(μ²ε)
Green–Naghdi
O(μ²)
Favrie–Gavrilyuk
O(μ²+μλ⁻¹)
Fully dispersive Saint-Venant
O(με)
Saint-Venant
O(μ)

A first method

Recall we (and [Lagrange,Boussinesq,Rayleigh]) had an expansion of the **Dirichlet-to-Neumann operator**

$$\mathcal{G}^{\mu}[\epsilon\zeta]\psi = -\mu\nabla\cdot\left(\int_{-1}^{\epsilon\zeta(t,\cdot)}\nabla_{\mathbf{x}}\Phi(\cdot,z)\,\mathrm{d}z\right)$$

with Φ solution to

$$\Phi + \mu\ell[\epsilon\zeta]\Phi = \psi, \qquad \ell[\epsilon\zeta]\Phi(\cdot,z) \stackrel{\text{def}}{=} -\int_{z}^{\epsilon\zeta}\int_{-1}^{z'}\Delta_{\mathbf{x}}\Phi(\cdot,z'')\,\mathrm{d}z''\,\mathrm{d}z'.$$

$$\Phi = \sum_{k=0}^{N}(-\mu\ell[\epsilon\zeta])^{k}\psi + \mathcal{O}(\mu^{N+1}).$$

This yields to **extended Green–Naghdi systems** [Matsuno '15,'16].

✕ The loss of derivatives is $2N+p$ for some p. No hope of convergence when $N\to\infty$, by the modal analysis.

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A second method

We have another expansion of the **Dirichlet-to-Neumann operator**

$$\mathcal{G}^{\mu}[\epsilon\zeta]\psi = \sum_{k=0}^{N} \epsilon^{k}\, \mathrm{d}^{k}\mathcal{G}^{\mu}[0](\zeta,\dots,\zeta)\psi + \mathcal{O}(\epsilon^{N+1}).$$

Plugging the truncated expansion into the Hamiltonian yields a hierarchy of models [Craig&Sulem '93] and also [Dommermuth&Yue '87, West et al. '87].

This is known as the **high-order spectral method**.

✓ The series converge (shape-analyticity of $\mathcal{G}^{\mu}$)
✗ Each of the models could be ill-posed [Ambrose,Bona&Nicholls '14]
✓ Well-posedness can be restored without any cost [VD&Melinand]

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The how-to guide to all (?) other methods

1. Select a variational formulation of the Laplace problem

$$\begin{cases} \mu\Delta_{\mathbf{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \{(\mathbf{x}, z),\ -1 < z < \epsilon\zeta(t, \mathbf{x})\}, \\ \Phi = \psi & \text{on } \{(\mathbf{x}, z),\ z = \epsilon\zeta(t, \mathbf{x})\}, \\ \partial_z\Phi = 0 & \text{on } \{(\mathbf{x}, z),\ z = -1\}. \end{cases}$$

2. Select a vertical distribution $\{\Psi_i(\mathbf{x}, z, \epsilon\zeta)\}_i$ and define the "finite-dimensional" vector space

$$V = \left\{\Phi,\ \Phi(t, \mathbf{x}, z) = \sum_{i=0}^{N} \phi_i(\mathbf{x}, t)\Psi_i(\mathbf{x}, z, \epsilon\zeta(t, \mathbf{x}))\right\}.$$

3. Define Φ_N^{app} as the Galerkin approximation of the variational problem.
4. Plug in the D2N operator, then the Hamiltonian.
5. Use Hamilton's equations and enjoy.

An example

$$V = \left\{ \Phi,\ \Phi(t,\mathbf{x},z) = \sum_{i=0}^{N} \phi_i(\mathbf{x},t)\Psi_i(\mathbf{x},z,\epsilon\zeta(t,\mathbf{x})) \right\}.$$

Setting $\Psi_i(\mathbf{x},z,\epsilon\zeta(t,\mathbf{x})) = (z+1)^{2i}$ (motivated by the [Boussinesq,Rayleigh] shallow-water expansion) yields the **Isobe–Kakinuma model**

$$\begin{cases} \partial_t\zeta + \sum_{i=0}^N \nabla\cdot\big(\frac{h^{2i+1}}{2i+1}\nabla\phi_i\big) = 0, \\ \partial_t\psi + \zeta + \epsilon(\sum_{i=0}^N 2ih^{2i}\phi_i)(\sum_{j=0}^N \nabla\cdot(\frac{h^{2j+1}}{2j+1}\nabla\phi_i) \\ \qquad + \frac{\epsilon}{2}\big(|\sum_{i=0}^N h^{2i}\nabla\phi_i|^2 + \frac{1}{\mu}(\sum_{i=0}^N 2ih^{2i-1}\phi_i)^2\big) = 0, \end{cases} \tag{IK}$$

with $h = 1+\epsilon\zeta$ and $\{\phi_i\}_{i\in\{0,1,\dots,N\}}$ solution to

$$\begin{cases} \sum_{j=0}^N \Big(-\frac{2i}{(2j+1)(2i+2j+1)}h^{2i+2j+1}\Delta\phi_j - \frac{1}{\mu}\frac{4ij}{2i+2j-1}h^{2i+2j-1}\phi_j\Big) = 0 \\ \qquad\qquad \forall i\in\{1,\dots,N\}, \\ \sum_{i=0}^N h^{2i}\phi_i = \psi. \end{cases}$$

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$$\boxed{\text{Isobe–Kakinuma} \;=\; (\text{ Hyperbolic }) \times (\text{ Elliptic }).}$$

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Other models with similar features can be derived with other choices for $\{\Psi_i(\mathbf{x}, z, \epsilon\zeta)\}_i$ [Athanassoulis&Belibassakis '99][Lynett&Liu '04][Klopman,vanGroesen&Dingemans '10].
Yet only the Isobe–Kakinuma model benefits from [Iguchi '18].

✓ full justification as a model of order $\mathcal{O}(\mu^{1+2N})$ (recall Padé approximants).

A brief introduction to interfacial waves

Waves at the interface between two homogeneous layers is a natural generalization of the water waves framework.
New phenomena arise.

- Role of the density contrast

  ⇝ Boussinesq approximation, rigid-lid framework [VD '14,'16]
- Kelvin–Helmholtz instabilities (KH)

  ⇝ ill-posedness (!)

Do shallow water models predict the propagation of sharp interfaces?

- ✓ The hydrostatic model (which extends the Saint-Venant model) *tames* KH. [Guyenne,Lannes&Saut '10][Bresch&Renardy '11]
  (WP when $h_1, h_2 > 0$ and $\gamma\epsilon|\mathbf{u}_1 - \mathbf{u}_2|^2 < a_0$ with some explicit $a_0(h_1, h_2) > 0$).
- ✕ The Miyata–Choi–Camassa model (which extends Green–Naghdi) *overestimates* KH. [Jo&Choi '02][Lannes&Ming '15] (modal analysis).
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What we have done

- Rigorously justified standard and widely used models for water waves (*Saint-Venant, Green–Naghdi*);
- Rigorously justified a hyperbolic relaxation of the Green–Naghdi system (*Favrie–Gavrilyuk*);
- Formally derived a class of high-order models, and rigorously justified a family (*Isobe–Kakinuma*);
- Ventured into the world of interfacial waves (*Choi–Camassa, Kakinuma*).

What we have not done

- Compared high-order models and their limit towards the water waves system;
- Entered the world of non-potential and/or continuously stratified flows;
- Said anything on solutions besides local-in-time existence (existence and stability of solitary waves, global existence vs finite-time singularity);
- Used models for practical problems (bottom reconstruction, fluid-structure interaction, *etc.*).

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Thank you for your attention