Modeling water waves and internal waves

Lecture 5 : interfacial waves

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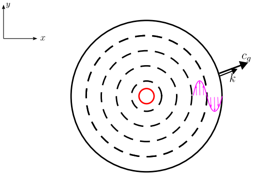
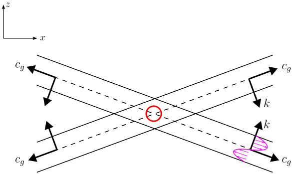
Qn : What are we looking at?

https://science.nasa.gov/earth/earth-observatory/internal-waves-in-the-indian-ocean-44567/

Internal waves, surface waves and interfacial waves

	(x, y) Surface waves	(x, z) Internal waves
Assumption	$\rho \equiv$ csst	$N^2 = \frac{\partial_z \rho}{\rho} \equiv$ csst
Dispersion relation	$\omega((k_x, k_y)) = \tilde{\omega}(\lvert(k_x, k_y)\rvert)$	$\omega((k_x, k_z)) = \tilde{\omega}(k_x/k_z)$
For given pulsation	Fixed wavevector **length**	Fixed wavevector **direction**
Group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$	**Parallel** to wavevector	**Orthogonal** to wavevector





Partially based on [Beckebanze, Raja & Maas (2019)]

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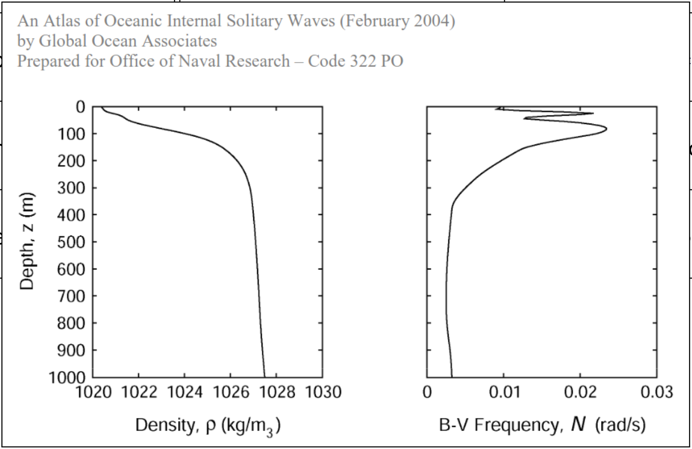
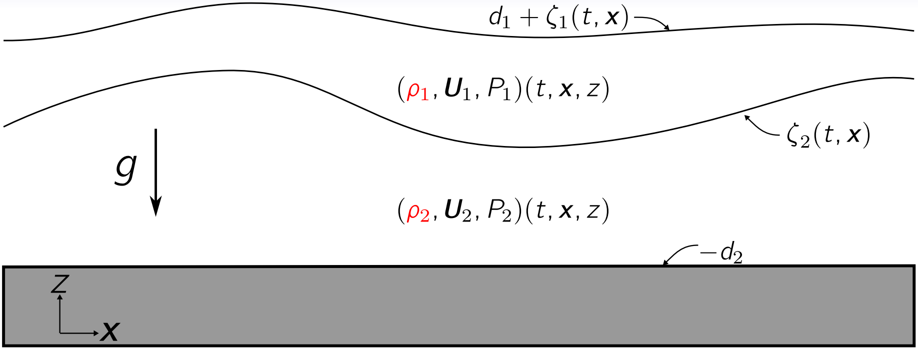


Density profile derived from SD2 data collected on 1 February 1972 at 7.000N, 94.970E
Source NODC Global Ocean Temperature and Salinity Profiles (Jun 1991) [Jackson&Apel 2004]

Outline

1. Interfacial waves
2. Kelvin–Helmholtz instabilities
3. Sharp stratification limit

Interfacial waves



Under the bilayer assumption for densities, and irrotationality in each layer, the dispersion relation of the free-surface/interface Euler equations reads

$$\omega(\mathbf{k})^4 - \frac{g|\mathbf{k}|\rho_2(\tanh(d_1|\mathbf{k}|)+\tanh(d_2|\mathbf{k}|))}{\rho_2+\rho_1\tanh(d_1|\mathbf{k}|)\tanh(d_2|\mathbf{k}|)}\omega(\mathbf{k})^2 + \frac{g^2|\mathbf{k}|^2(\rho_2-\rho_1)\tanh(d_1|\mathbf{k}|)\tanh(d_2|\mathbf{k}|)}{\rho_2+\rho_1\tanh(d_1|\mathbf{k}|)\tanh(d_2|\mathbf{k}|)} = 0.$$

$\rightsquigarrow$ **Two modes of propagation**

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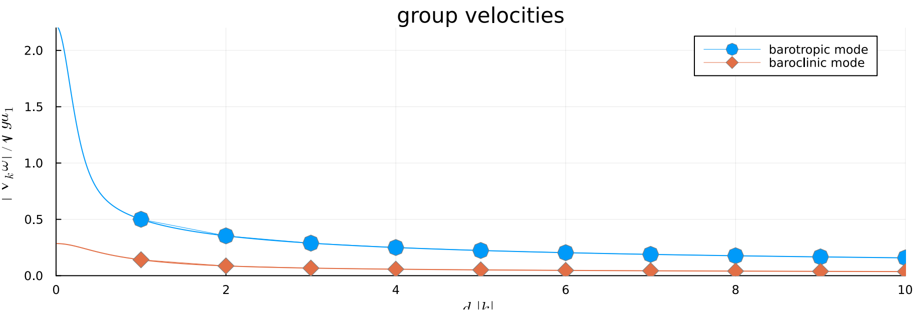
$\rightsquigarrow$ **Two modes of propagation**



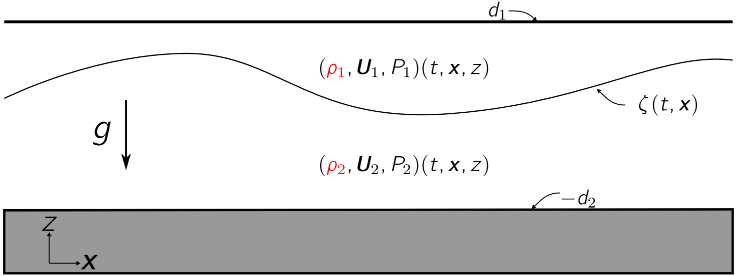
Qn : What happens as $\Delta\rho := 1 - \frac{\rho_1}{\rho_2} \to 0$?

The rigid-lid limit

As $\Delta\rho := 1 - \frac{\rho_1}{\rho_2} \to 0$, we observe a decoupling between the timescales of the two modes of propagation:

$$\begin{cases} \omega_+(\mathbf{k})^2 \to g|\mathbf{k}| \tanh((d_1 + d_2)|\mathbf{k}|) \\ \omega_-(\mathbf{k})^2 \sim g\Delta\rho|\mathbf{k}| \frac{\tanh(d_1|\mathbf{k}|)\tanh(d_2|\mathbf{k}|)}{\tanh(d_1|\mathbf{k}|)+\tanh(d_2|\mathbf{k}|)} \end{cases} \qquad \text{as } \Delta\rho \to 0.$$

- $\omega_+(\mathbf{k})$ corresponds to the homogeneous **free-surface** case.
- $\omega_-(\mathbf{k})$ corresponds to the bilayer **flat rigid-lid** case.



Outline

Kelvin–Helmholtz instabilities

Linearizing the flat rigid-lid system about a **shear-velocity solution** ($\mathbf{U}_x = \mathbf{U}_1$ in the upper layer and $\mathbf{U}_x = \mathbf{U}_2$ in the lower layer, and $\mathbf{U}_z = 0$) gives the dispersion relation

$$(\omega(\mathbf{k}) - \mathbf{c}(\mathbf{k}) \cdot \mathbf{k})^2 = a(\mathbf{k})b(\mathbf{k})$$

where

$$a(\mathbf{k}) := g\,\tfrac{\rho_2-\rho_1}{\rho_1+\rho_2} - \tfrac{\rho_1\rho_2/(\rho_1+\rho_2)}{\rho_2 \tanh(d_1|\mathbf{k}|)+\rho_1 \tanh(d_2|\mathbf{k}|)}\,\frac{((\mathbf{U}_2 - \mathbf{U}_1)\cdot \mathbf{k})^2}{|\mathbf{k}|},$$

$$b(\mathbf{k}) := \tfrac{(\rho_1+\rho_2)\tanh(d_1|\mathbf{k}|)\tanh(d_2|\mathbf{k}|)}{\rho_2 \tanh(d_1|\mathbf{k}|)+\rho_1 \tanh(d_2|\mathbf{k}|)}\,|\mathbf{k}|,$$

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Qn : What goes wrong?

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As soon as $\theta := (\mathbf{U}_2 - \mathbf{U}_1) \cdot \frac{\mathbf{k}}{|\mathbf{k}|} \neq 0$, $\mathrm{Im}(\omega)(\mathbf{k}) \neq 0$ if $|\mathbf{k}|$ is sufficiently large, and $\mathrm{Im}(\omega)(\mathbf{k}) \sim \frac{\sqrt{\rho_1\rho_2}}{\rho_1+\rho_2}|(\mathbf{U}_2-\mathbf{U}_1)\cdot\mathbf{k}|$ as $(\mathbf{U}_2-\mathbf{U}_1)\cdot\mathbf{k} \to \infty$.

$\rightsquigarrow$ **Kelvin–Helmholtz instabilities** Animation

Qn : Why are we seeing interfacial waves?

Interfacial tension

Adding **interface tension effect** and non-dimensionalizing equations yields

$$(\omega(\mathbf{k}) - \mathbf{c}(\mathbf{k})\cdot\mathbf{k})^2 = a(\mathbf{k})b(\mathbf{k})$$

where ($\gamma = \rho_1/\rho_2$, $\delta = d_1/d_2$)

$$a(\mathbf{k}) := (1 + \mathrm{Bo}|\mathbf{k}|^2) - \sqrt{\mu}\gamma\frac{1}{\tanh(\sqrt{\mu}\delta|\mathbf{k}|)+\gamma\tanh(\sqrt{\mu}|\mathbf{k}|)}\frac{(\epsilon(\mathbf{U}_2-\mathbf{U}_1)\cdot\mathbf{k})^2}{|\mathbf{k}|},$$

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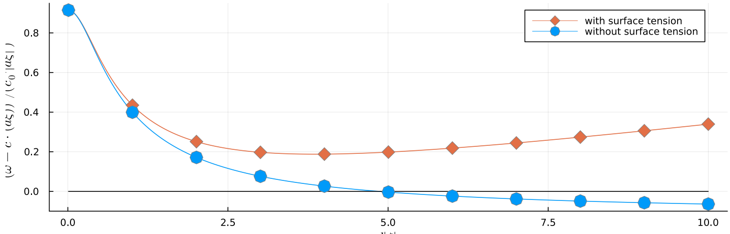
$\rightsquigarrow$ stability if $\Upsilon \stackrel{\text{def}}{=} \gamma^2\mu\epsilon^4/\mathrm{Bo}$ is sufficiently small.



Some mathematical results

Interfacial waves

[Ebin (1988), Iguchi,Tanaka&Tani (1997),Lebeau (2002), Kamotski&Lebeau (2005), Wu (2006)]

The initial-value problem for interfacial waves is strongly ill-posed.

[Ambrose&Masmoudi (2007), Cheng,Coutand&Shkoller (2008),Shatah&Zeng (2008,2011)]

The presence of interfacial tension restores well-posedness.

[Lannes (2013)]

Large-time well-posedness in finite-regularity spaces under a nonlinear criterion that is satisfied if $\Upsilon \stackrel{\text{def}}{=} \gamma^2\mu\epsilon^4/\mathrm{Bo}$ is sufficiently small.

Shallow-water models

[Ovsjannikov (1979), Armi (1986)]

Hyperbolicity and hence well-posedness in Sobolev spaces of the (rigid-lid or free-surface) **bilayer shallow water system** without interfacial tension if $\gamma\epsilon^2|\mathbf{u}_2 - \mathbf{u}_1|^2$ is sufficiently small.

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[D (2014,2016)]

Rigorous justification of the rigid-lid system as $\Delta\rho := 1 - \gamma \to 0$.

[D & Iguchi (2023,2024)]

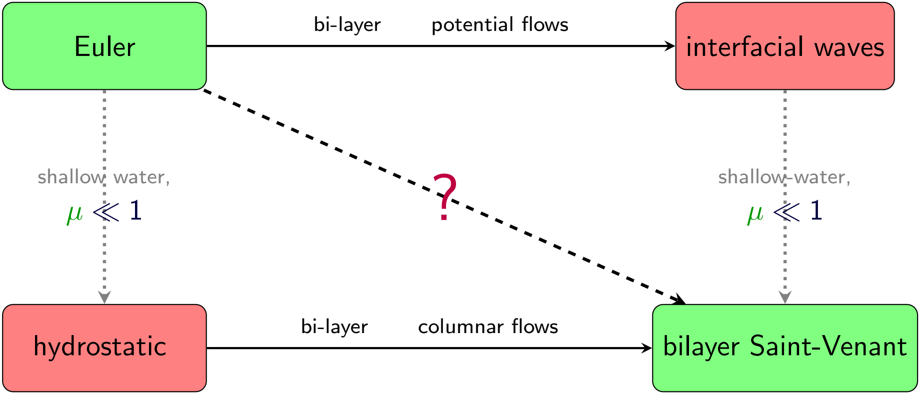
Well-posedness in Sobolev spaces of the **Kakinuma model** (without interfacial tension) under the sharp hyperbolicity criterion if

$$\frac{\gamma\epsilon^2}{(h_1+\gamma h_2)(2N+1)(N+1)}|\mathbf{u}_2 - \mathbf{u}_1|^2 \leq a =: 1 + \mathcal{O}(\epsilon),$$

where N is the rank of the model.

Qn : From a modeling viewpoint, are we satisfied?

A narrow path



Qn : Is it possible to justify the bilayer shallow-water system in a simultaneous bi-layer and $\mu \to 0$ limit?

Outline

The sharp stratification limit

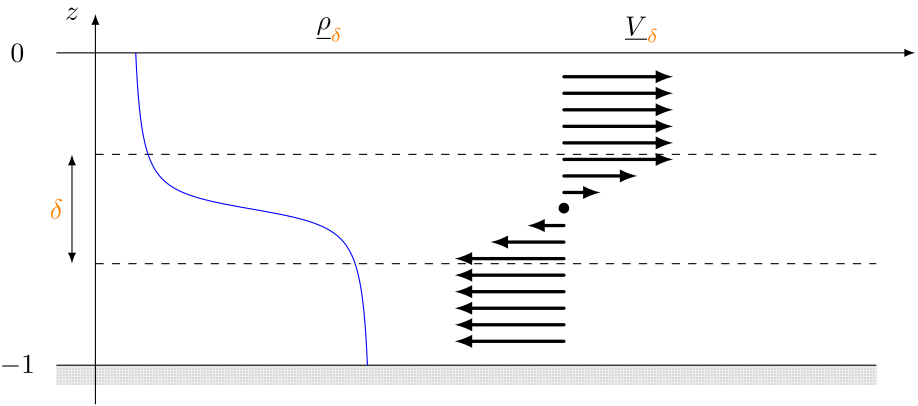
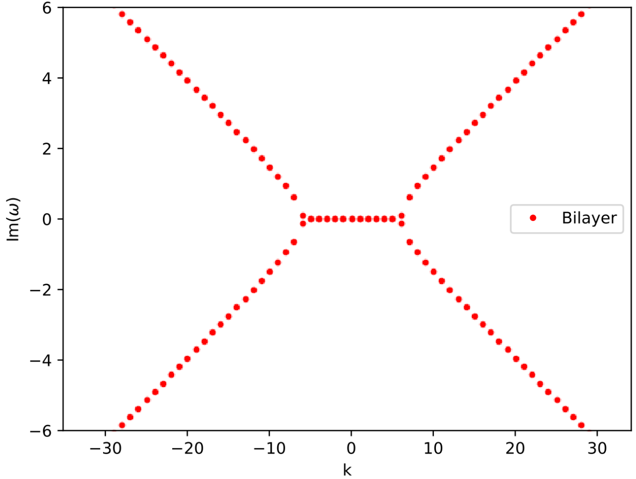


Figure by T. Fradin

Qn : Is it possible to justify the bilayer shallow-water system in a simultaneous $\delta \to 0$ and $\mu \to 0$ limit?

Sometimes the answer is "no"

Numerical computation of the dispersion relation of the stratified Euler equation in the limit $\delta \to 0$ [Fradin (2026)]



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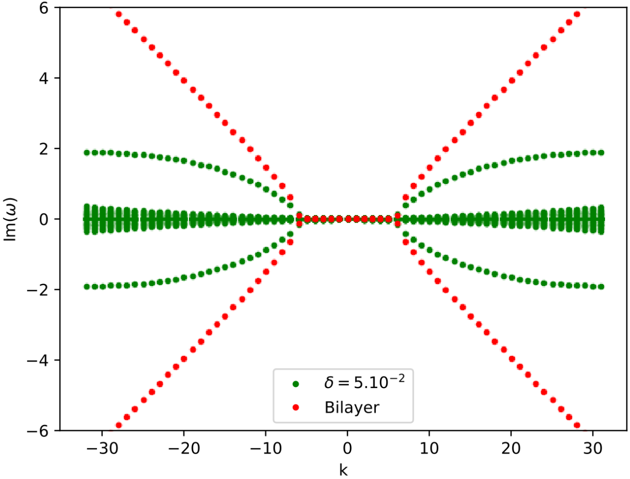

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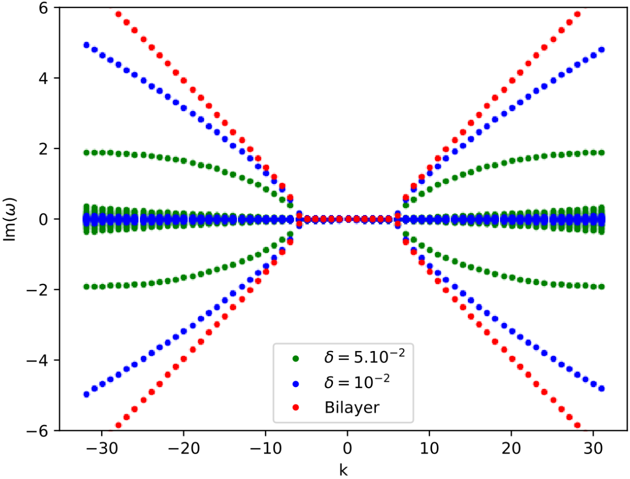

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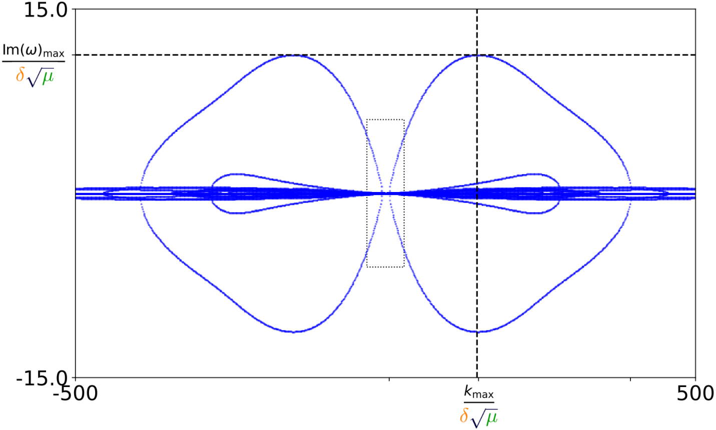

$\rightsquigarrow$amplification of the component with wavenumber $k_{\max}/(\delta\sqrt{\mu})$ on the timescale $T \approx \delta\sqrt{\mu}$

Qn : What modeling assumption is wrong ?

Gent & McWilliams parameterization

In geophysical applications one typically adds diffusive "turbulent" contributions from unresolved small-scale phenomena. One typical example is the [Gent&McWilliams (1990)] parameterization of eddy contributions.

In the context of the hydrostatic density-stratified equations this yieds

$$\begin{cases} \partial_t h + \partial_x\big(h(u+u_\star)\big) = 0, \\ \partial_t u + (u+u_\star)\partial_x u + M\partial_x h = 0, \end{cases} \qquad \text{with } u_\star := \kappa\frac{-\partial_x h}{h} \qquad (\mathrm{H}_\kappa)$$

where $(M\eta)(\cdot,r) := \rho(0)\int_0^1 \eta(\cdot,r')\mathrm{d}r' + \int_0^r \varrho'(r')\int_r^1 \eta(\cdot,r'')\mathrm{d}r''\mathrm{d}r'$.

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[Adim,Bianchini & D (2024)]

Consider a bi-layer profile satisfying the hyperbolicity criterion, and (h_δ^0, u_δ^0) regular initial data converging as $\delta \to 0$ towards the bi-layer profile.

For any $\kappa > 0$, (i) there exists solutions to (H_κ) and $(h_\delta, u_\delta)|_{t=0} = (h_\delta^0, u_\delta^0)$ defined on a time-interval independent of δ and (ii) these solutions (h_δ, u_δ) converge towards the corresponding solution to the bilayer shallow-water system with Gent&McWilliams contributions.

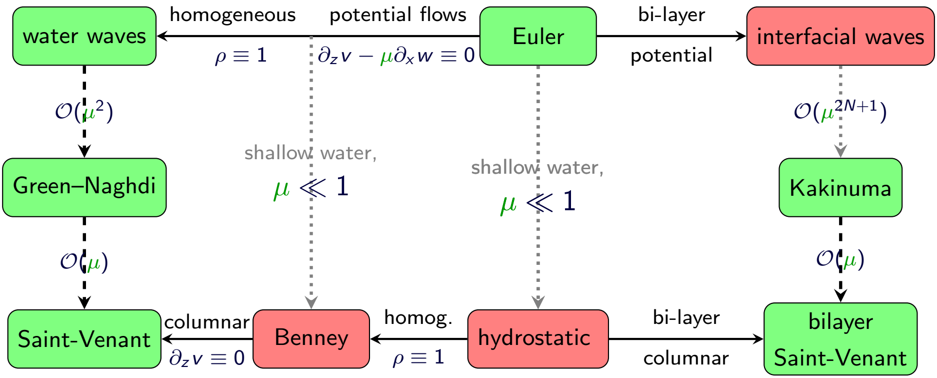
Thoughts to go

- The structure of the background profiles determines waves properties (internal, interfacial...)
- Interfacial waves propagate despite Kelvin–Helmholtz instabilities.
- The mathematical modeling of this problem is challenging, due to turbulent effects in the pycnocline.

Bibliography

On interfacial waves [D. Many models for Water Waves (2021)]
On internal waves [Fradin. Variable-density Euler equations and ocean dynamics (2026)]
and geophysical textbooks (see Lecture 4)

Summary of the models we met



Thank you for your attention!