Modeling water waves and internal waves

Lecture 4 : internal waves

Vincent Duchêne

Ravello School of Mathematical Physics

September 2026

Outline

Well-posedness?

The initial-value problem for the hydrostatic Euler equations in finite-regularity spaces is ill-posed in general, and open otherwise.

1. Vorticity
2. Density stratification
3. Non-hydrostatic pressure

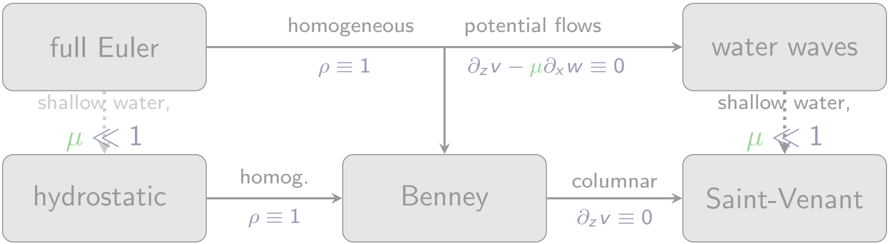
Meet Benney

The **hydrostatic**, **homogeneous** Euler equation with free surface (*a.k.a.* Benney system) reads in semi-Lagrangian coordinates ($d = 1$, $\epsilon = 1$)

$$\begin{cases} \partial_t h + v\partial_x h + h\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1], \\ \partial_t v + \int_0^1 (\partial_x h)(\cdot, r)\mathrm{d}r + v\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1]. \end{cases} \tag{Benney}$$

Notice if $h(\cdot, r) \equiv 1 + \zeta(\cdot)$ and $v(\cdot, r) \equiv u(\cdot)$ then (Benney) reduces to

$$\begin{cases} \partial_t \zeta + u\partial_x \zeta + (1+\zeta)\partial_x u = 0 & \text{in } \mathbb{R}, \\ \partial_t u + \partial_x \zeta + u\partial_x u = 0 & \text{in } \mathbb{R}. \end{cases} \tag{S.-V.}$$



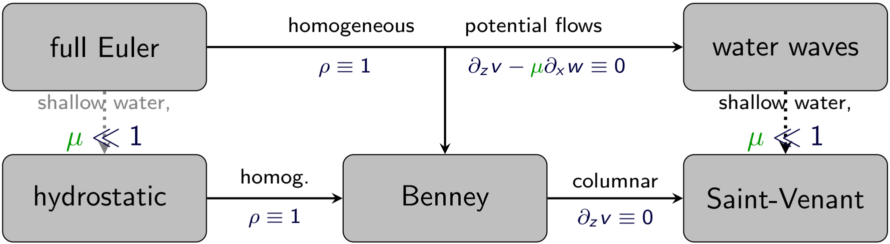
Meet Benney

The **hydrostatic**, **homogeneous** Euler equation with free surface (*a.k.a.* Benney system) reads in semi-Lagrangian coordinates ($d = 1, \epsilon = 1$)

$$\begin{cases} \partial_t h + v\partial_x h + h\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1], \\ \partial_t v + \int_0^1 (\partial_x h)(\cdot, r)\mathrm{d}r + v\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1]. \end{cases} \tag{Benney}$$

Notice if $h(\cdot, r) \equiv 1 + \zeta(\cdot)$ and $v(\cdot, r) \equiv u(\cdot)$ then (Benney) reduces to

$$\begin{cases} \partial_t \zeta + u\partial_x \zeta + (1+\zeta)\partial_x u = 0 & \text{in } \mathbb{R}, \\ \partial_t u + \partial_x \zeta + u\partial_x u = 0 & \text{in } \mathbb{R}. \end{cases} \tag{S.-V.}$$



Meet Benney

The **hydrostatic**, **homogeneous** Euler equation with free surface (*a.k.a.* Benney system) reads in semi-Lagrangian coordinates ($d = 1, \epsilon = 1$)

$$\begin{cases} \partial_t h + v\partial_x h + h\partial_x v = 0 & \text{in } \mathbb{R}\times[0,1], \\ \partial_t v + \int_0^1 (\partial_x h)(\cdot, r)\mathrm{d}r + v\partial_x v = 0 & \text{in } \mathbb{R}\times[0,1]. \end{cases} \quad \text{(Benney)}$$

Notice if $h(\cdot, r) \equiv 1 + \zeta(\cdot)$ and $v(\cdot, r) \equiv u(\cdot)$ then (Benney) reduces to

$$\begin{cases} \partial_t \zeta + u\partial_x \zeta + (1+\zeta)\partial_x u = 0 & \text{in } \mathbb{R}, \\ \partial_t u + \partial_x \zeta + u\partial_x u = 0 & \text{in } \mathbb{R}. \end{cases} \quad \text{(S.-V.)}$$

The energy method generally fails on (Benney) **Qn :** Why?

Meet Benney

The **hydrostatic**, **homogeneous** Euler equation with free surface (*a.k.a.* Benney system) reads in semi-Lagrangian coordinates ($d = 1, \epsilon = 1$)

$$\begin{cases} \partial_t h + v\partial_x h + h\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1], \\ \partial_t v + \int_0^1 (\partial_x h)(\cdot, r)\mathrm{d}r + v\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1]. \end{cases} \quad \text{(Benney)}$$

Notice if $h(\cdot, r) \equiv 1 + \zeta(\cdot)$ and $v(\cdot, r) \equiv u(\cdot)$ then (Benney) reduces to

$$\begin{cases} \partial_t \zeta + u\partial_x \zeta + (1+\zeta)\partial_x u = 0 & \text{in } \mathbb{R}, \\ \partial_t u + \partial_x \zeta + u\partial_x u = 0 & \text{in } \mathbb{R}. \end{cases} \quad \text{(S.-V.)}$$

The energy method generally fails on (Benney) because for given $(h, v)(r)$ the operator $L_{v,h} := \begin{pmatrix} v & h \\ \int_0^1 \bullet & v \end{pmatrix}$ cannot be "symmetrized".

$\rightsquigarrow$ There are deep connections between well-posedness/ill-posedness issues and the spectral stability of shear velocity solutions, *i.e.* $\mathrm{Spec}(L_{v,h}) \subset \mathbb{R}$.

Spectral stability

[Di Martino, El Hassanieh, Godlewski, Guillod & Sainte-Marie (2025)]

Under some regularity assumptions on v one has

1. $\mathrm{Spec}\left(\begin{pmatrix} v & 1 \\ \int_0^1 \bullet & v \end{pmatrix}\right) = v([0,1]) \bigcup \mathrm{Spec}_{\mathrm{d}}$, where

$$\mathrm{Spec}_{\mathrm{d}} = \left\{ c \in \mathbb{C} \setminus v([0,1]) \ : \ \int_0^1 (c - v(r))^{-2} \mathrm{d}r = 1 \right\}.$$

2. $\mathrm{Spec}_{\mathrm{d}} \cap \mathbb{R} = \{c_-, c_+\}$ where $c_- \in [\inf(v) - 1, \inf(v)]$ and $c_+ \in [\sup(v), \sup(v) + 1]$.
3. $\mathrm{Spec}_{\mathrm{d}} \subset \mathbb{R}$ if
   (i) for all $r \in [0,1]$ $v'(r) \neq 0$ and $v''(r) \neq 0$ [Rayleigh]; or
   (ii) for all $r \neq r_\star$, $v'(r) < 0$ and $v''(r)(r - r_\star) \leq 0$ [Fjørtoft]
4. $\mathrm{Spec}_{\mathrm{d}} \not\subset \mathbb{R}$ if $v(r_\star) = 0$ and v^{-2} is integrable. [Renardy (2009)]

Qn : Do spectral stability imply well-posedness, and conversely?

Spectral stability

[Di Martino, El Hassanieh, Godlewski, Guillod & Sainte-Marie (2025)]

Under some regularity assumptions on v one has

1. $\operatorname{Spec}\left(\begin{pmatrix} v & 1 \\ \int_0^1 \bullet & v \end{pmatrix}\right) = v([0,1]) \bigcup \operatorname{Spec}_{\mathrm{d}}$, where
$$\operatorname{Spec}_{\mathrm{d}} = \left\{ c \in \mathbb{C} \setminus v([0,1]) \ : \ \int_0^1 (c - v(r))^{-2} \mathrm{d}r = 1 \right\}.$$
2. $\operatorname{Spec}_{\mathrm{d}} \cap \mathbb{R} = \{c_-, c_+\}$ where
$c_- \in [\inf(v) - 1, \inf(v)]$ and $c_+ \in [\sup(v), \sup(v) + 1]$.
3. $\operatorname{Spec}_{\mathrm{d}} \subset \mathbb{R}$ if
(i) for all $r \in [0,1]$ $v'(r) \neq 0$ and $v''(r) \neq 0$ [Rayleigh]; or
(ii) for all $r \neq r_\star$, $v'(r) < 0$ and $v''(r)(r - r_\star) \leq 0$ [Fjørtoft]
4. $\operatorname{Spec}_{\mathrm{d}} \not\subset \mathbb{R}$ if $v(r_\star) = 0$ and v^{-2} is integrable. [Renardy (2009)]

Qn : Do spectral stability imply well-posedness, and conversely?

Spectral stability

[Di Martino, El Hassanieh, Godlewski, Guillod & Sainte-Marie (2025)]

Under some regularity assumptions on v one has

1. $\mathrm{Spec}\left(\begin{pmatrix} v & 1 \\ \int_0^1 \bullet & v \end{pmatrix}\right) = v([0,1]) \bigcup \mathrm{Spec}_{\mathrm{d}}$, where

$$\mathrm{Spec}_{\mathrm{d}} = \left\{ c \in \mathbb{C} \setminus v([0,1]) \ : \ \int_0^1 (c - v(r))^{-2} \mathrm{d}r = 1 \right\}.$$

2. $\mathrm{Spec}_{\mathrm{d}} \cap \mathbb{R} = \{c_-, c_+\}$ where
   $c_- \in [\inf(v) - 1, \inf(v)]$ and $c_+ \in [\sup(v), \sup(v) + 1]$.
3. $\mathrm{Spec}_{\mathrm{d}} \subset \mathbb{R}$ if
   (i) for all $r \in [0,1]$ $v'(r) \neq 0$ and $v''(r) \neq 0$ [Rayleigh]; or
   (ii) for all $r \neq r_\star$, $v'(r) < 0$ and $v''(r)(r - r_\star) \leq 0$ [Fjørtoft]
4. $\mathrm{Spec}_{\mathrm{d}} \not\subset \mathbb{R}$ if $v(r_\star) = 0$ and v^{-2} is integrable. [Renardy (2009)]

Qn : Do spectral stability imply well-posedness, and conversely?

Spectral stability

[Di Martino, El Hassanieh, Godlewski, Guillod & Sainte-Marie (2025)]

Under some regularity assumptions on v one has

1. $\mathrm{Spec}\left(\begin{pmatrix} v & 1 \\ \int_0^1 \bullet & v \end{pmatrix}\right) = v([0,1]) \bigcup \mathrm{Spec}_{\mathrm{d}}$, where
$$\mathrm{Spec}_{\mathrm{d}} = \left\{ c \in \mathbb{C} \setminus v([0,1]) \ : \ \int_0^1 (c - v(r))^{-2} \mathrm{d}r = 1 \right\}.$$
2. $\mathrm{Spec}_{\mathrm{d}} \cap \mathbb{R} = \{c_-, c_+\}$ where
$c_- \in [\inf(v) - 1, \inf(v)]$ and $c_+ \in [\sup(v), \sup(v) + 1]$.
3. $\mathrm{Spec}_{\mathrm{d}} \subset \mathbb{R}$ if
(i) for all $r \in [0,1]$ $v'(r) \neq 0$ and $v''(r) \neq 0$ [Rayleigh]; or
(ii) for all $r \neq r_\star$, $v'(r) < 0$ and $v''(r)(r - r_\star) \leq 0$ [Fjørtoft]
4. $\mathrm{Spec}_{\mathrm{d}} \not\subset \mathbb{R}$ if $v(r_\star) = 0$ and v^{-2} is integrable. [Renardy (2009)]

Qn : Do spectral stability imply well-posedness, and conversely?

Spectral stability *vs* well-posedness

Spectral stability does not imply well-posedness (in general) but

- [Teshukov (1992)] formally argues that (if $\underline{v}' \neq 0$) spectral stability implies that (Benney) is diagonalizable and hence well-posed.
- [Grenier (1999), Brenier (1999), Masmoudi&Wong (2012)] proved well-posedness (in the rigid-lid framework) under Rayleigh criterion.

Spectral instability implies ill-posedness[1] for (Benney) [Han-Kwan&Nguyen (2016)]

Formal argument. Notice that $(h, v)(t, x, r) = (1, \underline{v}(r)) + \varepsilon(\tilde{h}, \tilde{v})(t, x, r)$ with $(\tilde{h}, \tilde{v})$ solution to

$$\partial_t \begin{pmatrix} \tilde{h} \\ \tilde{v} \end{pmatrix} + L_{\underline{v}} \partial_x \begin{pmatrix} \tilde{h} \\ \tilde{v} \end{pmatrix} = 0. \qquad \text{(lin-Benney)}$$

satisfies (Benney) up to $\mathcal{O}(\varepsilon^2)$ terms.

If there exists $(\dot{h}, \dot{v})$ and $c \in \mathbb{C} \setminus \mathbb{R}$ such that $L_{\underline{v}} \begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix} = c \begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix}$, then for all $k \in \mathbb{R}$,

$(\tilde{h}, \tilde{v})(t, x, r) := (\dot{h}, \dot{v})(r) e^{ik(x-ct)} + \text{c.c.}$ satisfies (lin-Benney) and

$$\left\| (\tilde{h}, \tilde{v}) \right\|_{L^\infty} \approx e^{k|\mathrm{Im}(c)|t}.$$

[1] the solution map is not (Hölder) continuous with respect to initial data.

Spectral stability *vs* well-posedness

Spectral stability does not imply well-posedness (in general) but

- [Teshukov (1992)] formally argues that (if $v' \neq 0$) spectral stability implies that (Benney) is diagonalizable and hence well-posed.
- [Grenier (1999), Brenier (1999), Masmoudi&Wong (2012)] proved well-posedness (in the rigid-lid framework) under Rayleigh criterion.

Spectral instability implies ill-posedness[1] for (Benney) [Han-Kwan&Nguyen (2016)]

Formal argument. Notice that $(h, v)(t, x, r) = (1, \underline{v}(r)) + \varepsilon(\tilde{h}, \tilde{v})(t, x, r)$ with $(\tilde{h}, \tilde{v})$ solution to

$$\partial_t \begin{pmatrix} \tilde{h} \\ \tilde{v} \end{pmatrix} + L_{\underline{v}} \partial_x \begin{pmatrix} \tilde{h} \\ \tilde{v} \end{pmatrix} = 0. \tag{lin-Benney}$$

satisfies (Benney) up to $\mathcal{O}(\varepsilon^2)$ terms.

If there exists $(\dot{h}, \dot{v})$ and $c \in \mathbb{C} \setminus \mathbb{R}$ such that $L_{\underline{v}} \begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix} = c \begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix}$, then for all $k \in \mathbb{R}$,

$(\tilde{h}, \tilde{v})(t, x, r) := (\dot{h}, \dot{v})(r) e^{ik(x-ct)} + \text{c.c.}$ satisfies (lin-Benney) and

$$\left\| (\tilde{h}, \tilde{v}) \right\|_{L^\infty} \approx e^{k|\mathrm{Im}(c)|t}.$$

[1] the solution map is not (Hölder) continuous with respect to initial data.

Outline

Well-posedness?

The initial-value problem for the hydrostatic Euler equations in finite-regularity spaces is ill-posed in general, and open otherwise.

1. Vorticity
2. Density stratification
3. Non-hydrostatic pressure

Meet density stratification

The **hydrostatic**, **density-stratified** Euler equation with free surface reads in semi-Lagrangian coordinates ($d = 1, \epsilon = 1$)

$$\begin{cases} \partial_t h + v\partial_x h + h\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1], \\ \partial_t v + \frac{1}{\varrho} M\partial_x h + v\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1]. \end{cases} \tag{hydro}$$

where $(M\eta)(\cdot, r) := \rho(0) \int_0^1 \eta(\cdot, r')\mathrm{d}r' + \int_0^r \varrho'(r') \int_r^1 \eta(\cdot, r'')\mathrm{d}r''\mathrm{d}r'$.

Qn : What does “density stratification” means?

Well-posedness?

The initial-value problem for (hydro) in finite-regularity spaces is open.

Meet density stratification

The **hydrostatic**, **density-stratified** Euler equation with free surface reads in semi-Lagrangian coordinates ($d = 1, \epsilon = 1$)

$$\begin{cases} \partial_t h + v\partial_x h + h\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1], \\ \partial_t v + \frac{1}{\varrho} M\partial_x h + v\partial_x v = 0 & \text{in } \mathbb{R} \times [0,1]. \end{cases} \tag{hydro}$$

where $(M\eta)(\cdot, r) := \rho(0) \int_0^1 \eta(\cdot, r')\mathrm{d}r' + \int_0^r \varrho'(r') \int_r^1 \eta(\cdot, r'')\mathrm{d}r''\mathrm{d}r'$.

Qn : What does "density stratification" means?

Well-posedness?

The initial-value problem for (hydro) in finite-regularity spaces is open.

Spectral analysis

Given $(h(r), \varrho(r), v(r))$ we seek information on Spec $\left(\begin{pmatrix} v & h \\ \frac{1}{\varrho}M & v \end{pmatrix}\right)$ where $(M\eta)(\cdot, r) := \rho(0)\int_0^1 \eta(\cdot, r')\mathrm{d}r' + \int_0^r \varrho'(r')\int_r^1 \eta(\cdot, r'')\mathrm{d}r''\mathrm{d}r'$.

$$\begin{pmatrix} v & h \\ \frac{1}{\varrho}M & v \end{pmatrix}\begin{pmatrix} \eta \\ u \end{pmatrix} = c\begin{pmatrix} \eta \\ u \end{pmatrix} \Leftrightarrow -\left(\frac{\varrho}{h}(v-c)^2\zeta'\right)' + \varrho'\zeta = 0, \quad \zeta(r) := \int_0^r \eta(r')\mathrm{d}r'$$

which is (equivalent to) the **Taylor–Goldstein equation**.

Qn : What physical quantity does ζ represent?

Spectral analysis

Given $(h(r), \varrho(r), v(r))$ we seek information on Spec $\left(\begin{pmatrix} v & h \\ \frac{1}{\varrho}M & v \end{pmatrix}\right)$ where $(M\eta)(\cdot, r) := \rho(0)\int_0^1 \eta(\cdot, r')\mathrm{d}r' + \int_0^r \varrho'(r')\int_r^1 \eta(\cdot, r'')\mathrm{d}r''\mathrm{d}r'$.

$$\begin{pmatrix} v & h \\ \frac{1}{\varrho}M & v \end{pmatrix}\begin{pmatrix} \eta \\ u \end{pmatrix} = c\begin{pmatrix} \eta \\ u \end{pmatrix} \Leftrightarrow -\left(\frac{\varrho}{h}(v-c)^2\zeta'\right)' + \varrho'\zeta = 0, \quad \zeta(r) := \int_0^r \eta(r')\mathrm{d}r'$$

which is (equivalent to) the **Taylor–Goldstein equation**.

Special situations

1. Homogeneous case
   If $\varrho \equiv 1$ we recover the condition $\int_0^1 h(r)(c - v(r))^{-2}\mathrm{d}r = \int_0^1 h(r)\mathrm{d}r$.
2. No-shear case
   If $v \equiv 0$ and $\varrho' < 0$ then we have a **Sturm–Liouville problem**.
   $\rightsquigarrow$ there exists solutions $c_n^2 \approx \frac{1}{n^2}$ $(n \in \mathbb{N})$ and ζ_n with n nodes.

Spectral analysis

Given $(h(r), \varrho(r), v(r))$ we seek information on $\mathrm{Spec}\left(\begin{pmatrix} v & h \\ \frac{1}{\varrho}M & v \end{pmatrix}\right)$ where

$$(M\eta)(\cdot, r) := \rho(0)\int_0^1 \eta(\cdot, r')\mathrm{d}r' + \int_0^r \varrho'(r')\int_r^1 \eta(\cdot, r'')\mathrm{d}r''\mathrm{d}r'.$$

$$\begin{pmatrix} v & h \\ \frac{1}{\varrho}M & v \end{pmatrix}\begin{pmatrix} \eta \\ u \end{pmatrix} = c\begin{pmatrix} \eta \\ u \end{pmatrix} \Leftrightarrow -\left(\frac{\varrho}{h}(v-c)^2\zeta'\right)' + \varrho'\zeta = 0, \quad \zeta(r) := \int_0^r \eta(r')\mathrm{d}r'$$

which is (equivalent to) the **Taylor–Goldstein equation**.

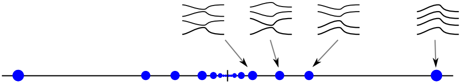
Special situations

1. Homogeneous case

   If $\varrho \equiv 1$ we recover the condition $\int_0^1 h(r)(c - v(r))^{-2}\mathrm{d}r = \int_0^1 h(r)\mathrm{d}r$.
2. No-shear case

   If $v \equiv 0$ and $\varrho' < 0$ then we have a **Sturm–Liouville problem**.

   $\rightsquigarrow$ there exists solutions $c_n^2 \approx \frac{1}{n^2}$ $(n \in \mathbb{N})$ and ζ_n with n nodes.



Spectral analysis

Given $(h(r), \varrho(r), v(r))$ we seek information on $c \in \mathbb{C}$ and $\zeta(r)$ satisfying

$$-\left(\frac{\varrho}{h}(v-c)^2\zeta'\right)' + \varrho'\zeta = 0.$$

Under some regularity assumptions on (h, ϱ, v) one has

1. Assuming that for all $r \in [0,1]$, $\varrho(r) > 0, \varrho'(r) > 0$ and $\frac{1}{4}v'(r)^2 \leq \frac{h(r)\varrho'(r)}{\varrho(r)}$ there is no solution $c \in \mathbb{C} \setminus \mathbb{R}$. [Miles , Howard (1961)]
2. In this case, there exists solutions $c_n \notin v([0,1])$ $(n \in \mathbb{N})$ which are approximated as $c_n \sim \{\inf v, \sup v\}$ by solutions to
$$\tan\left(\int_0^1 \frac{I(r)}{v(r)-c}\mathrm{d}r\right) = \frac{-(v(0)-c)I(0)}{h(0)-(v(0)-c)v'(0)/2}$$
where $I(r) = \sqrt{\frac{h(r)\varrho'(r)}{\varrho(r)} - \frac{v'(r)^2}{4}}$.
3. If in addition $v'(r) \neq 0$ for all $r \in [0,1]$, this yields
$$|c_n - u(\circ)| \sim C_\circ \exp\left(-\frac{|v'(\circ)|}{I(\circ)}n\pi\right), \quad \circ \in \{0,1\}, n \to \infty.$$

Spectral analysis

Given $(h(r), \varrho(r), v(r))$ we seek information on $c \in \mathbb{C}$ and $\zeta(r)$ satisfying

$$-\Big(\frac{\varrho}{h}(v-c)^2\zeta'\Big)' + \varrho'\zeta = 0.$$

Under some regularity assumptions on (h, ϱ, v) one has

1. Assuming that for all $r \in [0,1]$, $\varrho(r) > 0$, $\varrho'(r) > 0$ and $\frac{1}{4}v'(r)^2 \leq \frac{h(r)\varrho'(r)}{\varrho(r)}$ there is no solution $c \in \mathbb{C} \setminus \mathbb{R}$. [Miles , Howard (1961)]
2. In this case, there exists solutions $c_n \notin v([0,1])$ $(n \in \mathbb{N})$ which are approximated as $c_n \sim \{\inf v, \sup v\}$ by solutions to
$$\tan\Big(\int_0^1 \frac{I(r)}{v(r)-c}\mathrm{d}r\Big) = \frac{-(v(0)-c)I(0)}{h(0)-(v(0)-c)v'(0)/2}$$
where $I(r) = \sqrt{\frac{h(r)\varrho'(r)}{\varrho(r)} - \frac{v'(r)^2}{4}}$.
3. If in addition $v'(r) \neq 0$ for all $r \in [0,1]$, this yields
$$|c_n - u(\circ)| \sim C_\circ \exp\Big(-\frac{|v'(\circ)|}{I(\circ)}n\pi\Big), \quad \circ \in \{0,1\}, n \to \infty.$$

Spectral analysis

Given $(h(r),\varrho(r),v(r))$ we seek information on $c\in\mathbb{C}$ and $\zeta(r)$ satisfying

$$-\Big(\frac{\varrho}{h}(v-c)^2\zeta'\Big)'+\varrho'\zeta=0.$$

Under some regularity assumptions on (h,ϱ,v) one has

1. Assuming that for all $r\in[0,1]$, $\varrho(r)>0,\varrho'(r)>0$ and $\frac14 v'(r)^2\leq\frac{h(r)\varrho'(r)}{\varrho(r)}$ there is no solution $c\in\mathbb{C}\setminus\mathbb{R}$. [Miles , Howard (1961)]
2. In this case, there exists solutions $c_n\notin v([0,1])$ $(n\in\mathbb{N})$ which are approximated as $c_n\sim\{\inf v,\sup v\}$ by solutions to
$$\tan\Big(\int_0^1\frac{I(r)}{v(r)-c}\mathrm{d}r\Big)=\frac{-(v(0)-c)I(0)}{h(0)-(v(0)-c)v'(0)/2}$$
where $I(r)=\sqrt{\frac{h(r)\varrho'(r)}{\varrho(r)}-\frac{v'(r)^2}{4}}$.
3. If in addition $v'(r)\neq0$ for all $r\in[0,1]$, this yields
$$|c_n-u(\circ)|\sim C_\circ\exp\Big(-\frac{|v'(\circ)|}{I(\circ)}n\pi\Big),\quad\circ\in\{0,1\},n\to\infty.$$

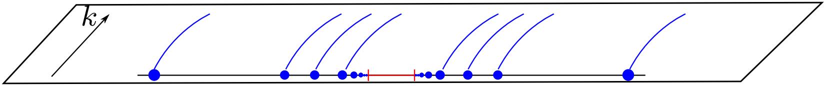
Outline

Removing the hydrostatic assumption

If we re-introduce the non-hydrostatic contributions then an additional term adds to the **Taylor–Goldstein equation**.

Some consequences are:

- Branches $c_n(k_x)$ emerge from eigenvalues of the hydrostatic problems for sufficiently small values of the horizontal frequency k_x.



Qn : Are you surprised?

- For sufficiently large values of the horizontal frequency k_x, $c_n(k_x) \in \mathbb{R}$.

Qn : Are you surprised?

- If $v' \equiv 0$ and $\frac{h\varrho'}{\varrho} \equiv N^2$ (N is the Brunt-Väisälä frequency) and in the rigid-lid framework then

$$\omega_n^2(k_x) := k_x^2 c_n(k_x)^2 = \frac{N^2\,k_x^2}{k_z^2 + \mu^2 k_x^2}, \quad k_z = n\pi,\ n \in \mathbb{N}.$$

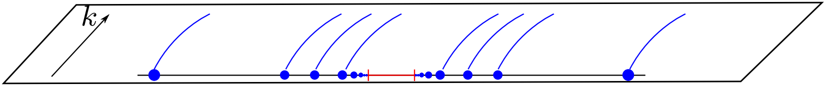
Qn : Are you surprised?

Removing the hydrostatic assumption

If we re-introduce the non-hydrostatic contributions then an additional term adds to the **Taylor–Goldstein equation**.

Some consequences are:

- Branches $c_n(k_x)$ emerge from eigenvalues of the hydrostatic problems for sufficiently small values of the horizontal frequency k_x.



  Qn : Are you surprised?

- For sufficiently large values of the horizontal frequency k_x, $c_n(k_x) \in \mathbb{R}$.

  Qn : Are you surprised?

- If $v' \equiv 0$ and $\frac{h\varrho'}{\varrho} \equiv N^2$ (N is the Brunt-Väisälä frequency) and in the rigid-lid framework then

$$\omega_n^2(k_x) := k_x^2 c_n(k_x)^2 = \frac{N^2\, k_x^2}{k_z^2 + \mu^2 k_x^2}, \quad k_z = n\pi, \;\; n \in \mathbb{N}.$$

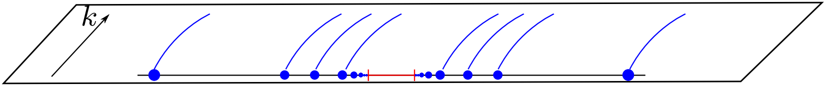
  Qn : Are you surprised?

Removing the hydrostatic assumption

If we re-introduce the non-hydrostatic contributions then an additional term adds to the **Taylor–Goldstein equation**.

Some consequences are:

- Branches $c_n(k_x)$ emerge from eigenvalues of the hydrostatic problems for sufficiently small values of the horizontal frequency k_x.



  Qn : Are you surprised?

- For sufficiently large values of the horizontal frequency k_x, $c_n(k_x) \in \mathbb{R}$.

  Qn : Are you surprised?

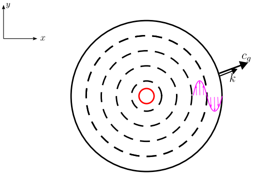
- If $v' \equiv 0$ and $\frac{h\varrho'}{\varrho} \equiv N^2$ (N is the Brunt-Väisälä frequency) and in the rigid-lid framework then

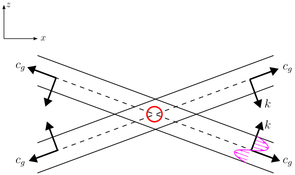
$$\omega_n^2(k_x) := k_x^2 c_n(k_x)^2 = \frac{N^2\, k_x^2}{k_z^2 + \mu^2 k_x^2}, \quad k_z = n\pi,\ n \in \mathbb{N}.$$

  Qn : Are you surprised?

Internal waves vs surface waves

	(x, y) Surface waves	(x, z) Internal waves
Dispersion relation	$\omega((k_x, k_y)) = \tilde{\omega}(\lvert(k_x, k_y)\rvert)$	$\omega((k_x, k_z)) = \tilde{\omega}(k_x/k_z)$
For given pulsation	Fixed wavevector **length**	Fixed wavevector **direction**
Group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$	**Parallel** to wavevector	**Orthogonal** to wavevector





Partially based on [Beckebanze, Raja & Maas (2019)]

Thoughts to go

- The situation with shear velocities and density variations is <u>much</u> richer than the irrotational, homogeneous (water-wave) case.
- As a rule of thumb, in our problems, shear velocity had a destabilizing effect and increasing-with-depth density a stabilizing effect.
- Internal waves are quite different from surface (or acoustic) waves.
- From a mathematical viewpoint, spectral stability and well-posedness are connected, but not equivalent.

Bibliography

[Gill. Atmosphere-ocean dynamics (1982)]

[Vallis. Atmospheric and oceanic fluid dynamics (2006)]

[Bühler, Waves and mean flows (2009)]

[Cushman-Roisin and Beckers. Introduction to geophysical fluid dynamics (2011)]

Next (and final) talk

V. Interfacial waves *vs* internal waves (and discussion)