# Modeling water waves and internal waves

## Lecture 3 : singular limits

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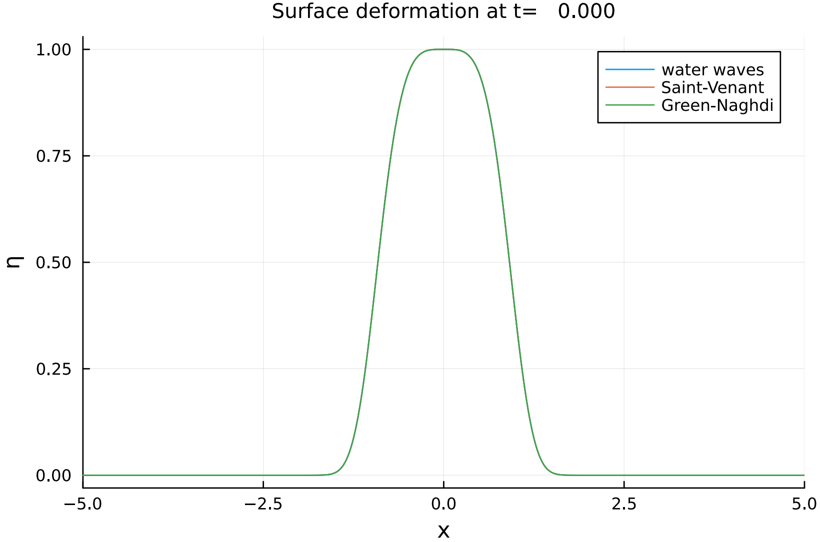
Ravello School of Mathematical Physics

September 2026

# Outline


1. Motivation
2. First singular phenomenon
3. Second singular phenomenon
4. Back to water waves

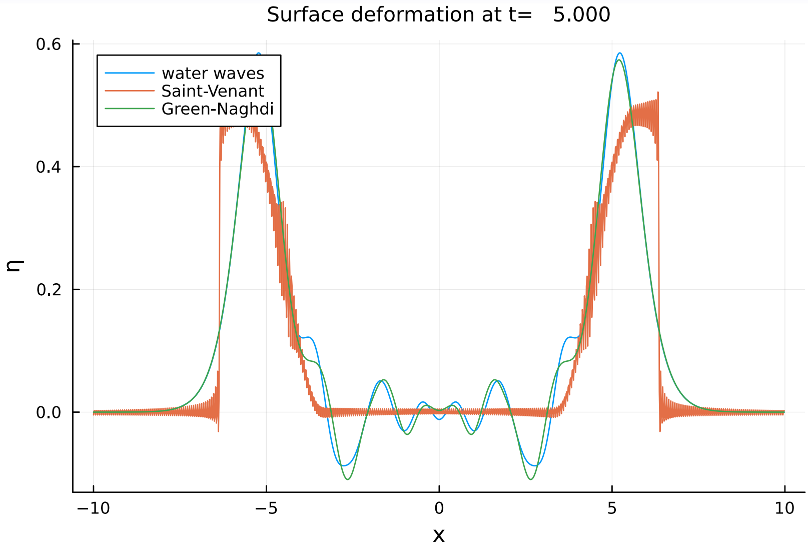
# Hydrostatic vs non-hydrostatic




Animation

$$\mu = 0.1, \ \epsilon = 0.25$$

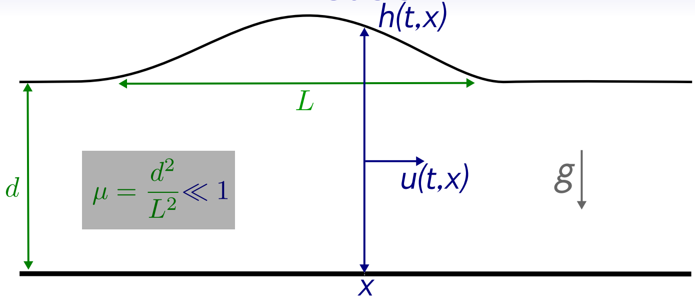
# Hydrostatic vs non-hydrostatic




Computation time: 6 sec. (water-waves), 2 sec. (shallow water), 30 sec. (Green–Naghdi).

**Qn :** what is wrong here?

# Models




The shallow water model is (with notation $h = 1 + \epsilon\zeta$, and setting $\epsilon = 1$)

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ h\partial_t u + h\partial_x h + hu\partial_x u = 0. \end{cases} \tag{SV}$$

The Green-Naghdi (dispersive) model is

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \mathfrak{T}^\mu[h]\partial_t u + h\partial_x h + hu\partial_x u - \frac{\mu}{3}\partial_x(h^3 u\partial_x^2 u - h^3(\partial_x u)^2) = 0, \end{cases} \tag{GN}$$

where $\mathfrak{T}^\mu[h] : v \mapsto hv - \frac{\mu}{3}\partial_x(h^3\partial_x v)$.

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## Rigorous justification

Consistency+stability estimates $\Longrightarrow$ Rigorous justification when $\mu \ll 1$

[Li (2006)][Alvarez-Samaniego & Lannes (2008)][Fujiwara & Iguchi (2015)]

**Qn :** Can you spot the issue ?

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**Qn :** Can you spot the issue ?

## Numerical approximation

Requires to invert $\mathfrak{T}^\mu[h]$ for (standard) numerical approximations.

# The "hyperbolization" trick

- The Green-Naghdi system can be reformulated as

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(\frac{1}{2}h^2 + hu^2) + \frac{\mu}{3}\partial_x(h^2\ddot{h}) = 0, \end{cases} \qquad \text{(GN)}$$

where $\dot{h} = \partial_t h + u\partial_x h$ and $\ddot{h} = \partial_t \dot{h} + u\partial_x \dot{h}$.

- This can again be reformulated as

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$\rightsquigarrow$ balance laws + $\boxed{\textit{constraint}}$

- Strategy [Favrie&Gavrilyuk(2017)][Escalante, Dumbser & Castro(2019)][Richard(2021)]
Relax the constraint by replacing the last equation with

$$\varepsilon^2 h(\partial_t p + u\partial_x p) = -w - h\partial_x u.$$

(**Qn :** Why specifically this?) and consider the singular limit $\varepsilon \to 0$.

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# A three-scale singular limit

After rescaling, the dimensionless relaxed Green-Naghdi system reads

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(\frac{1}{2}h^2 + hu^2) + \frac{\delta}{3\varepsilon}\partial_x(hp) = 0, \\ h(\partial_t w + u\partial_x w) = \frac{1}{\varepsilon}p, \\ h(\partial_t p + u\partial_x p) + \frac{\delta}{\varepsilon}h\partial_x u = -\frac{1}{\varepsilon}w. \end{cases} \tag{rGN}$$

where $\delta = \sqrt{\mu}$ is the shallow water parameter. We have $0 < \varepsilon \leq \delta \ll 1$.

After further manipulations, the system can (almost) be written as

$$\boxed{S_0(U)\partial_t U + S_1(U)\partial_x U + G(U) = \frac{\delta}{\varepsilon}\mathrm{L}\partial_x U + \frac{1}{\varepsilon}\mathrm{J}\, U}$$

with $S_0(U)$ sym. def. pos., $S_1(U)$ and L symmetric, J skew-symmetric.

- This strategy is currently developed within the Tolosa library https://tolosa-project.com

**Qn :** Do you know some problems that have a similar form?

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# Toy model: inviscid Burgers

Consider the inviscid Burgers equation

$$\partial_t u + \frac{1}{\varepsilon}(1+u)\partial_x u = 0.$$

then **Qn :** what happens?

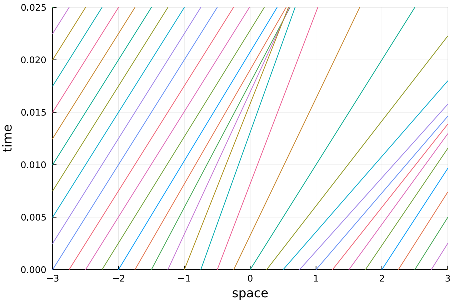
# Toy model: inviscid Burgers

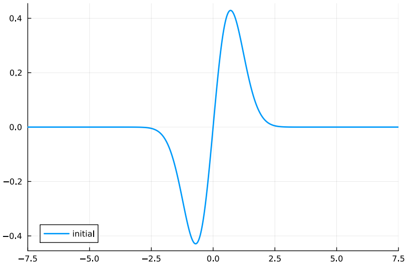
Consider the inviscid Burgers equation

$$\partial_t u + \frac{1}{\varepsilon}(1+u)\partial_x u = 0.$$

then we have a **gradient catastrophe** at time $t = \varepsilon(-\inf_{\mathbb{R}} \partial_x u|_{t=0})^{-1}$.

$\rightsquigarrow$ uniform control only if $\partial_x u|_{t=0} = \mathcal{O}(\varepsilon)$, *i.e.* **well-prepared** init. data.




Animation

# Prototypical example: shallow-water

Consider the limit $\varepsilon = \epsilon \to 0$ in the shallow-water equations

$$\begin{cases} \varepsilon\partial_t\zeta + \nabla_{\mathbf{x}} \cdot \big((1+\epsilon\zeta)\mathbf{u}\big) = 0 & \text{in } \mathbb{R}^d, \\ \varepsilon\partial_t\mathbf{u} + \epsilon(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}}\zeta = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{S-V}$$

**Qn :** Under which condition on initial data may I expect uniform-in-$\varepsilon$ control in $W^1_{t,\mathbf{x}}([0,T]\times\mathbb{R}^d)$?

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## Proposition [Klainerman&Majda '81 & '82][Schochet '86]

**For all** $T > 0$ there exists $\epsilon_0 > 0$ such that for all $\varepsilon = \epsilon \leq \epsilon_0$ and $(\zeta^0, \mathbf{u}^0)$ such that $\inf_{\mathbb{R}^d} 1 + \epsilon\zeta^0 \geq h_\star > 0$, $\big|(\zeta^0, \mathbf{u}^0)\big|_{H^s} \lesssim 1$, $\big|(\zeta^0, \nabla_{\mathbf{x}} \cdot \mathbf{u}^0)\big|_{H^{s-1}} \lesssim \varepsilon$ (with $s \geq 1 + d/2$),
then $(\zeta_\varepsilon, \mathbf{u}_\varepsilon) \in C([0,T]; H^s)$ solution to (S-V) and $(\zeta_\varepsilon, \mathbf{u}_\varepsilon)|_{t=0} = (\zeta^0, \mathbf{u}^0)$ satisfies

$$(\zeta_\varepsilon, \mathbf{u}_\varepsilon) \to (0, \mathbf{u}) \quad \text{in } C([0,T]; H^{s-1})$$

where $\mathbf{u}$ satisfies $\mathbf{u}|_{t=0} = \mathbf{u}^0$ and

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0 & \text{in } \mathbb{R}^d, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}} p = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{r.l.}$$

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**Qn :** What about **ill-prepared** initial data?

# Prototypical example: shallow-water

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**Proposition** [Klainerman&Majda '82]

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$$(\zeta_\varepsilon, \mathbf{u}_\varepsilon) \rightharpoonup (0,\mathbf{u}) \quad \textbf{weakly} \text{ in } L^\infty([0,T];H^s)$$

where $\mathbf{u}$ satisfies

$$\begin{cases} \nabla_{\mathbf{x}}\cdot\mathbf{u} = 0 & \text{in } \mathbb{R}^d, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}} p = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{r.l.}$$

**Qn :** Do we have $\mathbf{u}|_{t=0} = \mathbf{u}^0$?

# Prototypical example: shallow-water

**Proposition** [Klainerman&Majda '82]

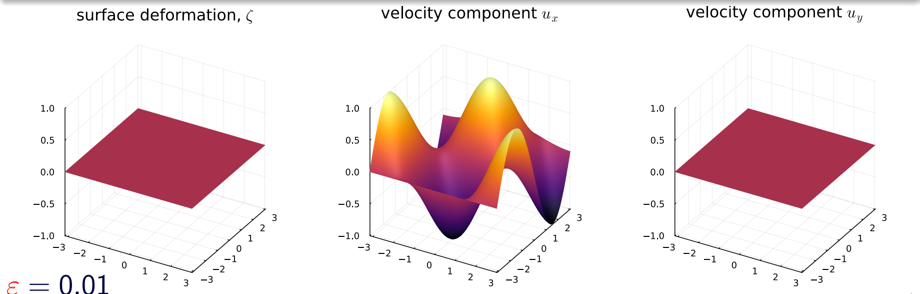
Solutions $(\zeta_\varepsilon, \mathbf{u}_\varepsilon) \in C([0,T]; H^s)$ to

$$\begin{cases} \varepsilon\partial_t\zeta_\varepsilon + \nabla_{\mathbf{x}} \cdot \big((1+\varepsilon\zeta_\varepsilon)\mathbf{u}_\varepsilon\big) = 0 & \text{in } \mathbb{R}^d, \\ \varepsilon\partial_t\mathbf{u}_\varepsilon + \varepsilon(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u}_\varepsilon + \nabla_{\mathbf{x}}\zeta_\varepsilon = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{S-V}$$

satisfy

$$(\zeta_\varepsilon, \mathbf{u}_\varepsilon) \rightharpoonup (0, \mathbf{u}) \quad \textbf{weakly in } L^\infty([0,T]; H^s)$$

where $\mathbf{u}$ satisfies $\nabla_{\mathbf{x}} \cdot \mathbf{u} = 0$ and $\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}} p = 0$ in $\mathbb{R}^d$



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# Another toy model

Consider the following toy model

$$\begin{cases} \partial_t h = 0, \\ h\partial_t u = \frac{1}{\varepsilon}\mathrm{i}u. \end{cases} \tag{toy-2}$$

Then **Qn :** what happens?

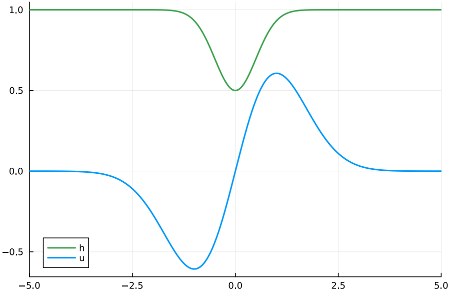
# Another toy model

Consider the following toy model

$$\begin{cases} \partial_t h = 0, \\ h\partial_t u = \frac{1}{\varepsilon}\mathrm{i}u. \end{cases} \tag{toy-2}$$

Then we have a the emergence of **rapid spatial oscillations**

$\rightsquigarrow$ uniform control in $W^{k,\infty}$ at time $t = 1$ only iff $\partial_x u|_{t=0} = \mathcal{O}(\varepsilon^k)$, i.e. $\partial_t^j u|_{t=0} = \mathcal{O}(1)$ for $j = 0, 1, \ldots, k$ (**strongly well-prepared** init. data).




Animation

# Another toy model

Consider the three-scale toy model

$$\begin{cases} \partial_t h = 0, \\ h\partial_t u = -\frac{\delta}{\varepsilon}\partial_x u + \frac{1}{\varepsilon}\mathrm{i}u. \end{cases} \tag{toy-3}$$

Then... **Qn :** what happens when $0 \ll \varepsilon \ll \delta \ll 1$?

# Another toy model

Consider the three-scale toy model

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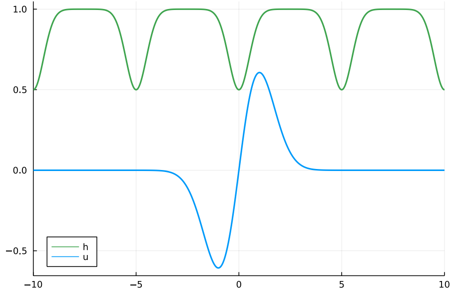
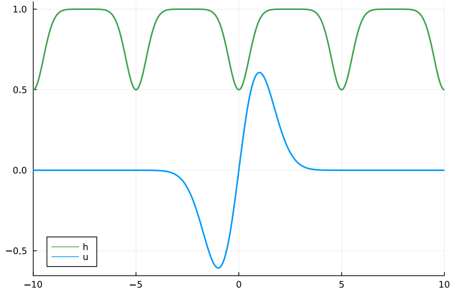
Then we have a the emergence of **new spatial scale of size $\mathcal{O}(\delta)$.**




$\varepsilon = 0.01$, $\delta = 0.05$ Animation




$\varepsilon = 0.01$, $\delta = 0.25$ Animation

**Qn :** Under which condition on initial data may I expect uniform-in-$(\delta, \varepsilon)$ control in $W^1_{t,\mathbf{x}}([0, T] \times \mathbb{R}^d)$?

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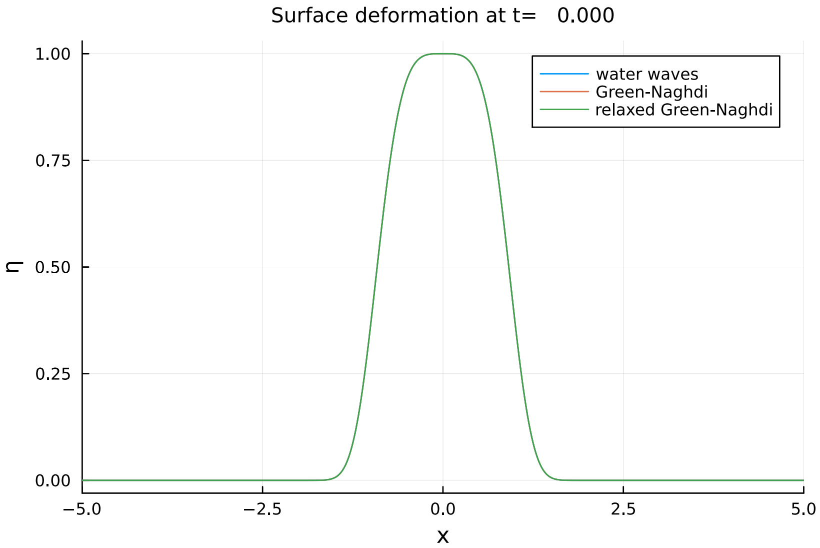
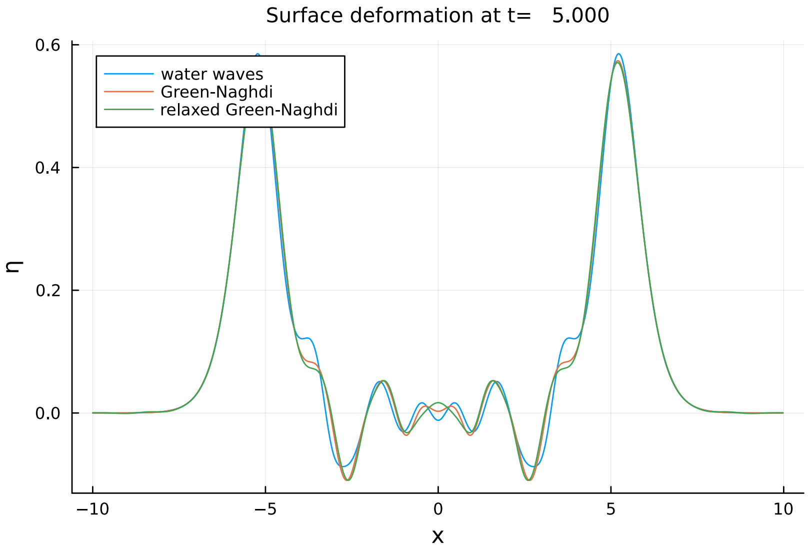
where $0 < \varepsilon \leq \delta \ll 1$.

# Numerical illustrations




Animation **Qn :** what do we observe?

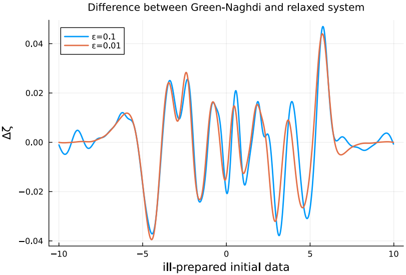
# Numerical illustrations



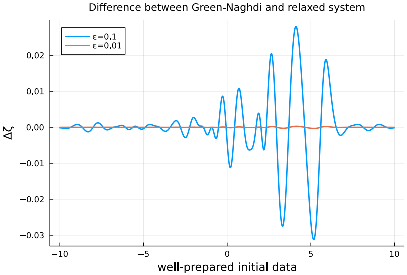

Animation **Qn :** what do we observe?

# Numerical illustrations

Preparing the initial data is crucial.




ill-prepared initial data




(mildly) well-prepared initial data

## Remarks

1. Preparing the initial data is not restrictive: we impose only initial data for the additional unknowns $(w, p)$ but not physical unknowns $(h, u)$.
2. We could consider strongly well-prepared initial data, but computations become very cumbersome.

# Control of an energy

**Proposition** [D, Duran, Msheik (2026)]

The control at time $t = 0$ of the energy functional[a]

$$\mathcal{E}_U(t) := \sum_{0 \leq i+j \leq s} \alpha_{j,i}^{-1} \left\| \partial_t^j \partial_x^i U(t,\cdot) \right\|_{L^2}$$

is propagated for positive times (uniformly with respect to $0 < \epsilon \leq \delta \leq 1$) for **admissible** sets of weights $\alpha_{j,i}$ .

[a] associated with solutions to the system

$$S_0(U)\partial_t U + S_1(U)\partial_x U + G(U) = \frac{\delta}{\epsilon} \mathrm{L} \partial_x U + \frac{1}{\epsilon} \mathrm{J}\, U.$$

with $S_0(U)$ sym. def. pos., $S_1(U)$ and $\mathrm{L}$ symmetric, $\mathrm{J}$ skew-symmetric.

**Rk:** The weight $\alpha_{j,i}$ measures how large $\left\| \partial_t^j \partial_x^i U(t,\cdot) \right\|_{L^2}$ is allowed to be. For instance, in order to ensure a control of $\left\| U \right\|_{W^{1,\infty}_{t,x}}$, we require

$$\alpha_{0,i} = \mathcal{O}(1),\ (0 \leq i \leq s_0) \quad \alpha_{1,i} = \mathcal{O}(1),\ (0 \leq i \leq s_0 - 1) \qquad (s_0 > 1 + d/2).$$

# Admissible weights (examples)

$$\mathcal{E}_U(t) := \sum_{0\le i+j\le s} \alpha_{j,i}^{-1} \big\|\partial_t^j \partial_x^i U(t,\cdot)\big\|_{L^2}.^{1}$$

**Ex 1:** $\alpha_{j,i} = 1$ for all $0 \le i + j \le s$ is admissible, but this is imposes a very strong restriction on the initial data.

[1] associated with solutions to the system

$$S_0(U)\partial_t U + S_1(U)\partial_x U + G(U) = \frac{\delta}{\epsilon}\mathsf{L}\partial_x U + \frac{1}{\epsilon}\mathsf{J}\,U.$$

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# Admissible weights (examples)

$$\mathcal{E}_U(t) := \sum_{0\leq i+j\leq s} \alpha_{j,i}^{-1} \|\partial_t^j \partial_x^i U(t,\cdot)\|_{L^2}.$$[1]

**Ex 1:** $\alpha_{j,i} = 1$ for all $0 \leq i+j \leq s$ is admissible, but this is imposes a very strong restriction on the initial data.

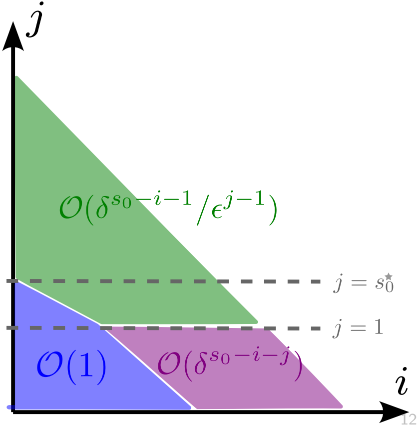
**Ex 2:** $\alpha_{j,i} = \max(\{1, \delta^{s_0-i-j}, \frac{\delta^{s_0-1-i}}{\varepsilon^{j-1}}\})$

Requires to satisfy

$$\|\partial_t^j U\|_{H^{s_0-j}} = \mathcal{O}(1)$$

(only) for all $j$ such that

$0 \leq j \leq \boxed{1 + (s_0-1)\frac{\log \delta}{\log \varepsilon}}$.



# Strategy

We seek (uniform) *a priori* estimates for energy functionals

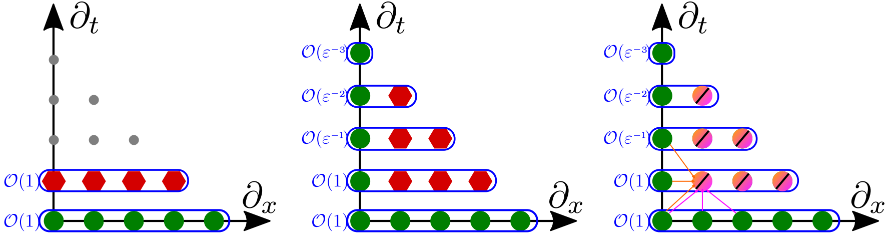
$$\mathcal{E}(t) := \sum_{0 \le i+j \le s} \alpha_{j,i}^{-1} \left\| \partial_t^j \partial_x^i U(t,\cdot) \right\|_{L^2}.$$

associated with solutions to the system

$$S_0(U)\partial_t U + S_1(U)\partial_x U + G(U) = \frac{\delta}{\varepsilon} \mathsf{L} \partial_x U + \frac{1}{\varepsilon} \mathsf{J}\, U.$$

with $S_0(U)$ sym. def. pos., $S_1(U)$ and $\mathsf{L}$ symmetric, $\mathsf{J}$ skew-symmetric.

**Sketch of the proof** : We follow the strategy of [Schochet '86] ($\mathsf{J} = 0$) which seeks to bootstrap the control of different components of the energy functional




By energy estimates By energy estimates By system of equations

# Main result

[D, Duran, Msheik]

Under a **scale-separation assumption** between groups of eigenvalues of the symbol of $\frac{\delta}{\varepsilon}\mathsf{L}\partial_x + \frac{1}{\varepsilon}\mathsf{J}$, the strategy à la Schochet works out provided that for some $s_0 > d/2 + 1$

- H1 $\alpha_{0,s_0} = \alpha_{1,s_0-1} = 1$,
- H2 $\alpha_{j,i+1} \geq \alpha_{j,i}$,
- H3 $\alpha_{j+1,i-1} \geq \alpha_{j,i}$ and $\alpha_{j+1,i-1} \leq \frac{\delta}{\varepsilon}\alpha_{j,i}$,
- H4 $\alpha_{1,i-1} \leq \alpha_{0,i}$,
- H5 $\prod_{k=0}^{\ell} \alpha_{j_k,i_k} \leq \alpha_{j^\star,i^\star} \times \alpha_{j_\star,i_\star}^{\ell}$ for all $(j_k, i_k)_{k=0,\dots,\ell}$, $(j^\star, i^\star)$, $(j_\star, i_\star)$ satisfying (i) and ( (iia) or (iib) ) where
  (i) $\sum j_k = j^\star + \ell j_\star$ and $\sum i_k = j^\star + \ell i_\star$,
  (iia) $j_\star \leq j_k \leq j^\star$ and $i_\star + j_\star \leq i_k + j_k \leq i^\star + j^\star$,
  (iib) $j_k = \theta_k j^\star + (1-\theta_k) j_\star$ and $i_k = \theta_k i^\star + (1-\theta_k) i_\star$ with $\theta_k \in [0, 1]$.

# Thoughts to go

- We have showed how **multi-scale singular limits** emerge in current strategies to simulate water waves.
- We have encountered some dangers of singular limits (wavebreaking, weak convergence, rapid oscillations).
- We have only scratched the surface of a very broad theory (in particular we did not discuss dispersive effects).

# Bibliography


- $\mathsf{J}=0$ (low Mach number, incompressible limit) [Klainerman&Majda, Browning&Kreiss, Schochet, Ukai...]
- $\mathsf{L}=0$ (rapid rotation, stiff relaxation) [Babin,Mahalov&Nicolaenko, Chemin,Desjardins, Gallagher&Grenier...]
- $\mathsf{J},\mathsf{L}\neq 0$ with prescribed limit ($\delta=\epsilon^\alpha$) [Feireisl&Novotný, Fanelli&Gallagher, Charve...]
- Three-scale problems ($\delta$ and $\epsilon$ independent) [Schochet et al.]


# Next talks

**IV.** We re-introduce vorticity and variable densities.
**V.** We move from internal waves to interfacial waves.