Modeling water waves and internal waves

Lecture 2 : shallow-water models

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Our aim

Approximate solutions to

$$\begin{cases} \partial_t \tilde\rho + \epsilon \mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde\rho + \epsilon w \partial_z \tilde\rho = 0 & \text{in } \Omega, \\ (1+\gamma\epsilon\tilde\rho)\big(\partial_t \mathbf{v} + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)\mathbf{v}\big) + \nabla_{\mathbf{x}} P = -\nabla_{\mathbf{x}}\zeta & \text{in } \Omega, \\ \mu(1+\gamma\epsilon\tilde\rho)\big(\partial_t w + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)w\big) + \partial_z P = -\gamma\tilde\rho & \text{in } \Omega, \\ \nabla_{\mathbf{x}}\cdot\mathbf{v} + \partial_z w = 0 & \text{in } \Omega, \end{cases} \qquad \text{(Euler)}$$

where $\Omega(t) := \{(\mathbf{x},z) \ : \ -1 < z < \epsilon\zeta(t,\mathbf{x})\}$ and

$$\begin{cases} w = 0 & \text{on } \{(\mathbf{x},z) \ : \ z = -1\}, \\ P = 0 & \text{on } \{(\mathbf{x},z) \ : \ z = \epsilon\zeta(t,\mathbf{x})\}, \\ \partial_t\zeta + \epsilon\mathbf{v}\cdot\nabla_{\mathbf{x}}\zeta - w = 0 & \text{on } \{(\mathbf{x},z) \ : \ z = \epsilon\zeta(t,\mathbf{x})\}. \end{cases} \qquad \text{(b.c.)}$$

with solutions to

$$\begin{cases} \partial_t\zeta + \nabla_{\mathbf{x}}\cdot\big((1+\epsilon\zeta)\mathbf{u}\big) = 0 & \text{in } \mathbb{R}^d, \\ \partial_t\mathbf{u} + \epsilon(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}}\zeta = 0 & \text{in } \mathbb{R}^d. \end{cases} \qquad \text{(S-V)}$$

Qn : What are benefits of the shallow-water system (S-V) w.r.t. (Euler)-(b.c.)?

Outline

"Water waves"

$$\begin{cases} \partial_t\tilde\rho + \epsilon\mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde\rho + \epsilon w\partial_z\tilde\rho = 0 & \text{in } \Omega, \\ (1+\gamma\epsilon\tilde\rho)\big(\partial_t\mathbf{v} + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)\mathbf{v}\big) + \nabla_{\mathbf{x}}P = -\nabla_{\mathbf{x}}\zeta & \text{in } \Omega, \\ \mu(1+\gamma\epsilon\tilde\rho)\big(\partial_t w + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)w\big) + \partial_z P = -\gamma\tilde\rho & \text{in } \Omega, \\ \nabla_{\mathbf{x}}\cdot\mathbf{v} + \partial_z w = 0 & \text{in } \Omega, \end{cases} \qquad \text{(Euler)}$$

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Step 1. Assume (**Qn :** Why can we?) that

$\tilde\rho = 0$ (homogeneous flow) and $\nabla_{\mathbf{x},z}\times(\mathbf{v},\mu w) = 0$ (irrotational flow).

Then $\boxed{(\mathbf{v},\mu w) =: \nabla_{\mathbf{x},z}\Phi}$ and...

Go to step 2

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The water waves system

We derived the water waves system (for homogeneous, irrotational flows)

$$\begin{cases} \partial_t\zeta + \epsilon(\nabla_{\mathbf{x}}\Phi)|_{z=\epsilon\zeta}\cdot\nabla_{\mathbf{x}}\zeta - \frac{1}{\mu}(\partial_z\Phi)|_{z=\epsilon\zeta} = 0 & \text{in } \mathbb{R}^d, \\ \partial_t\psi + \frac{\epsilon}{2}(|\nabla_{\mathbf{x},z}\Phi|^2)|_{z=\epsilon\zeta} + \zeta = 0 & \text{in } \mathbb{R}^d, \end{cases} \qquad \text{(water-waves)}$$

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Qn : What are benefits of (water-waves) with respect to (Euler)-(b.c.)?

Well-posedness [Alvarez-Samaniego–Lannes (2008), Iguchi (2009), Fradin (2026)]

There exists $T > 0$ **such that** for all (ζ^0, ψ^0) initial data of size 1 in a suitable Banach space X^s $(s \geq s_0)$,
there is a unique $(\zeta,\psi) \in C([0, T/\epsilon]; X^s)$ local-in-time solution to (water-waves)-(Laplace) and $\zeta|_{t=0} = \zeta^0$, $\psi|_{t=0} = \psi^0$.

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From the water waves to the shallow water system

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We wish to prove that solutions to (Laplace) satisfy

$$\epsilon(\nabla_{\mathbf{x}}\Phi)|_{z=\epsilon\zeta}\cdot\nabla_{\mathbf{x}}\zeta - \frac{1}{\mu}(\partial_z\Phi)|_{z=\epsilon\zeta} \approx \nabla_{\mathbf{x}}\cdot\big((1+\epsilon\zeta)\nabla_{\mathbf{x}}\psi\big)$$
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and then (water-waves) becomes (formally) (S-V) (with $\mathbf{u} = \nabla_{\mathbf{x}}\psi$).

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The Laplace problem

$$\begin{cases} \mu\Delta_{\mathbf{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega := \{(\mathbf{x},z) \ : \ -1 < z < \epsilon\zeta(t,\mathbf{x})\}, \\ \partial_z\Phi = 0 & \text{on } \{(\mathbf{x},z) \ : \ z = -1\}, \\ \Phi = \psi & \text{on } \{(\mathbf{x},z) \ : \ z = \epsilon\zeta(t,\mathbf{x})\}, \end{cases} \qquad \text{(Laplace)}$$

Proposition

Let $\zeta \in W^{1,\infty}(\mathbb{R}^d)$ be such that $\inf_{\mathbb{R}^d}(1+\epsilon\zeta) \geq h_\star > 0$. Then

1. For any $\psi \in \mathring{H}^1(\mathbb{R}^d)$ there exists a unique $\Phi - \psi \in H_0^1(\Omega)$ such that Φ is a variational solution to (Laplace).
2. There exists $p \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, if $(\nabla\psi, \zeta) \in H^{k+p}(\mathbb{R}^d)$ then $\nabla_{\mathbf{x},z}\Phi \in H^k(\Omega)$ and

$$\big\|\nabla_{\mathbf{x},z}\Phi\big\|_{H^k(\Omega)} \leq C(k, h_\star^{-1}, \big|\zeta\big|_{H^{k+p}(\mathbb{R}^d)})\big|\nabla\psi\big|_{H^{k+p}(\mathbb{R}^d)}$$

where $C(\bullet)$ depends non-decreasingly on $\bullet$, **but not on** $\mu \in (0,1]$.

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Sketch of the proof (**elliptic theory**; see *e.g.* [Lannes (2013)]).

1. weak formulation & Poincaré inequality (bounded domain in z and zero-trace at $z=\epsilon\zeta$)

   $\rightsquigarrow$ existence and uniqueness by Lax–Milgram.

2. Flattening the domain $(\widetilde{\mathbf{x}},\widetilde{z}) = (\mathbf{x}, \frac{z-\epsilon\zeta(x)}{1+\epsilon\zeta(x)})$,

$$\rightsquigarrow \begin{pmatrix} \sqrt{\mu}\nabla_{\widetilde{\mathbf{x}}} \\ \partial_{\widetilde{z}} \end{pmatrix} \cdot \left(\big(\mathrm{Id} + A(\epsilon\zeta, \mu\epsilon\nabla_{\mathbf{x}}\zeta)\big) \begin{pmatrix} \sqrt{\mu}\nabla_{\widetilde{\mathbf{x}}} \\ \partial_{\widetilde{z}} \end{pmatrix} \widetilde{\Phi} \right) = 0 \quad \text{in } \Sigma := \{(\mathbf{x},z) \ : \ -1 < z < 0\}.$$

Control $\mathbf{x}$-derivatives by finite induction on k, using product and commutator estimates in Sobolev spaces $H^k(\Sigma)$, and the variational bound in $L^2(\Sigma)$.

Control z-derivatives using directly the equation $\partial_z^2\Phi = -\mu\Delta_{\mathbf{x}}\Phi$.

Ex : Do it (or check details in [Lannes (2013), Chap. 2] or [D. (2021), Prop. 4.5]) □

Shallow-water asymptotics

$$\begin{cases} \mu\Delta_{\mathbf{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega := \{(\mathbf{x},z) \ : \ -1 < z < \epsilon\zeta(t,\mathbf{x})\}, \\ \partial_z\Phi = 0 & \text{on } \{(\mathbf{x},z) \ : \ z = -1\}, \\ \Phi = \psi & \text{on } \{(\mathbf{x},z) \ : \ z = \epsilon\zeta(t,\mathbf{x})\}, \end{cases} \qquad \text{(Laplace)}$$

Proposition (Consistency)

Let $\zeta \in W^{1,\infty}(\mathbb{R}^d)$ such that $\inf_{\mathbb{R}^d}(1+\epsilon\zeta) \geq h_\star > 0$. There exists $p \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, if $(\nabla\psi,\zeta) \in H^{k+p}(\mathbb{R}^d)$ then

$$\epsilon(\nabla_{\mathbf{x}}\Phi)|_{z=\epsilon\zeta} \cdot \nabla_{\mathbf{x}}\zeta - \frac{1}{\mu}(\partial_z\Phi)|_{z=\epsilon\zeta} = \nabla_{\mathbf{x}} \cdot \big((1+\epsilon\zeta)\nabla_{\mathbf{x}}\psi\big) + \mu R_1$$

$$\frac{\epsilon}{2}(|\nabla_{\mathbf{x},z}\Phi|^2)|_{z=\epsilon\zeta} = \frac{\epsilon}{2}(|\nabla_{\mathbf{x}}\psi|^2) + \mu R_2$$

with

$$\big|(R_1,\nabla_{\mathbf{x}}R_2)\big|_{H^k} \leq C(k, h_\star^{-1}, \big|(\zeta,\nabla\psi)\big|_{H^{k+p}})\big|(\zeta,\nabla\psi)\big|_{H^{k+p}}$$

Shallow-water asymptotics

$$\boxed{\begin{cases} \mu\Delta_{\mathbf{x}}\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega := \{(\mathbf{x},z)\ :\ -1 < z < \epsilon\zeta(t,\mathbf{x})\}, \\ \partial_z\Phi = 0 & \text{on } \{(\mathbf{x},z)\ :\ z = -1\}, \\ \Phi = \psi & \text{on } \{(\mathbf{x},z)\ :\ z = \epsilon\zeta(t,\mathbf{x})\}, \end{cases}} \qquad \text{(Laplace)}$$

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Shallow-water asymptotics

$$\Phi(\mathbf{x},z)=\psi(\mathbf{x})+\mu\int_z^{\zeta(\mathbf{x})}\int_{-1}^{z'}\Delta_{\mathbf{x}}\Phi(\mathbf{x},z'')\mathrm{d}z''$$

$$\rightsquigarrow \nabla_{\mathbf{x}}\Phi(\mathbf{x},z)=\nabla\psi(\mathbf{x})+\mathcal{O}(\mu),\quad \frac{1}{\mu}\partial_z\Phi(\mathbf{x},z)=(x+1)\Delta\psi(\mathbf{x})+\mathcal{O}(\mu)$$

Proposition (Consistency)

Let $\zeta\in W^{1,\infty}(\mathbb{R}^d)$ such that $\inf_{\mathbb{R}^d}(1+\epsilon\zeta)\geq h_\star>0$. There exists $p\in\mathbb{N}$ such that for all $k\in\mathbb{N}$, if $(\nabla\psi,\zeta)\in H^{k+p}(\mathbb{R}^d)$ then

$$\epsilon(\nabla_{\mathbf{x}}\Phi)|_{z=\epsilon\zeta}\cdot\nabla_{\mathbf{x}}\zeta-\frac{1}{\mu}(\partial_z\Phi)|_{z=\epsilon\zeta}=\nabla_{\mathbf{x}}\cdot\big((1+\epsilon\zeta)\nabla_{\mathbf{x}}\psi\big)+\mu R_1$$
$$\frac{\epsilon}{2}(|\nabla_{\mathbf{x},z}\Phi|^2)|_{z=\epsilon\zeta}=\frac{\epsilon}{2}(|\nabla_{\mathbf{x}}\psi|^2)+\mu R_2$$

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$$\big|(R_1,\nabla_{\mathbf{x}}R_2)\big|_{H^k}\leq C(k,h_\star^{-1},\big|(\zeta,\nabla\psi)\big|_{H^{k+p}})\big|(\zeta,\nabla\psi)\big|_{H^{k+p}}$$

Qn : Are we done yet? **Qn :** Can we (and should we) continue the

Stability

Consider the linearized shallow-water system

$$\begin{cases} \partial_t\dot\zeta + (1+\epsilon\bar\zeta)\nabla_{\mathbf{x}}\cdot\dot{\mathbf{u}} + \epsilon\bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\dot\zeta = 0 & \text{in } \mathbb{R}^d, \\ \partial_t\dot{\mathbf{u}} + \epsilon(\bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}})\dot{\mathbf{u}} + \nabla_{\mathbf{x}}\dot\zeta = 0 & \text{in } \mathbb{R}^d. \end{cases} \qquad \text{(lin. S-V)}$$

Proposition (Stability)

Given $(\dot\zeta, \dot{\mathbf{u}})$ and $(\bar\zeta, \bar{\mathbf{u}})$ sufficiently regular solutions to (lin. S-V) such that $\inf_{\mathbb{R}^d} 1+\epsilon\bar\zeta \geq h_\star > 0$, one has

$$\big|(\dot\zeta,\dot{\mathbf{u}})|_{t=T/\epsilon}\big|_{L^2(\mathbb{R}^d)} \leq C\big(h_\star^{-1}, \big|(\zeta,\bar{\mathbf{u}})\big|_{W^{1,\infty}_{t,\mathbf{x}}}, T\big)\big|(\dot\zeta,\dot{\mathbf{u}})|_{t=0}\big|_{L^2(\mathbb{R}^d)}$$

Sketch of the proof (**energy method**; see *e.g.* [Benzoni-Gavage&Serre (2007)]).

Integrate the first equation of (lin. S-V) against $\dot\zeta$, and the second equation against $(1+\epsilon\zeta)\dot{\mathbf{u}}$

$$\begin{aligned} \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\zeta,\psi) &= \epsilon\int_{\mathbb{R}^d}\frac{1}{2}(\nabla_{\mathbf{x}}\cdot\bar{\mathbf{u}})(\dot\zeta)^2 + \frac{1}{2}(\partial_t\bar\zeta + \nabla\cdot((1+\epsilon\bar\zeta)\bar{\mathbf{u}})|\dot{\mathbf{u}}|^2 - (\nabla\bar\zeta)\cdot(\dot\zeta\dot{\mathbf{u}})\mathrm{d}\mathbf{x} \\ &\leq \epsilon\, C\big(h_\star^{-1}, \big|(\zeta,\bar{\mathbf{u}})\big|_{W^{1,\infty}_{t,\mathbf{x}}}\big)\mathcal{E}(\zeta,\psi) \end{aligned}$$

where $\mathcal{E}(\zeta,\psi) := \frac{1}{2}\int_{\mathbb{R}^d}(\dot\zeta)^2 + (1+\epsilon\bar\zeta)|\dot{\mathbf{u}}|^2\mathrm{d}\mathbf{x}$. Then use Gronwall's Lemma. $\square$

Consistency + stability ⇒ convergence

Using the **stability estimate** with $(\dot\zeta,\dot{\mathbf{u}})=(\partial^k\zeta,\partial^k\dot{\mathbf{u}})$ allows to bootstrap a control in $H^s(\mathbb{R}^d)\subset W^{1,\infty}(\mathbb{R}^d)$ $(s\geq 1+d/2)$.

Well-posedness

There exists $T>0$ such that for all $(\zeta^0,\mathbf{u}^0)$ such that
$\inf_{\mathbb{R}^d} 1+\epsilon\zeta^0\geq h_\star>0$, and $\big|(\zeta^0,\mathbf{u}^0)\big|_{H^s}\lesssim 1 \qquad (s\geq 1+d/2)$,
there is a unique $(\zeta,\mathbf{u})\in C([0,T/\epsilon];H^s)$ solution to (S-V) and $(\zeta,\mathbf{u})|_{t=0}=(\zeta^0,\mathbf{u}^0)$.

Using the **stability estimate** with $(\dot\zeta,\dot{\mathbf{u}})$ the difference between the water waves and the shallow water solutions and the **consistency** yields

Convergence

There exists $p\in\mathbb{N}$ and $T>0$ such that for all k and all initial data (ζ^0,ψ^0) such that $\inf_{\mathbb{R}^d} 1+\epsilon\zeta^0\geq h_\star>0$ and $\big|(\zeta^0,\nabla\psi^0)\big|_{H^{k+p}}\lesssim 1$,
the solutions (ζ,ψ) to (water-waves) and $(\zeta_{\rm sw},\mathbf{u}_{\rm sw})$ to (S-V) satisfy

$$\forall\tau\in[0,T/\epsilon],\qquad \big|(\zeta-\zeta_{\rm sw},\nabla\psi-\mathbf{u}_{\rm sw})|_{t=\tau}\big|_{H^k}\lesssim \mu\tau.$$

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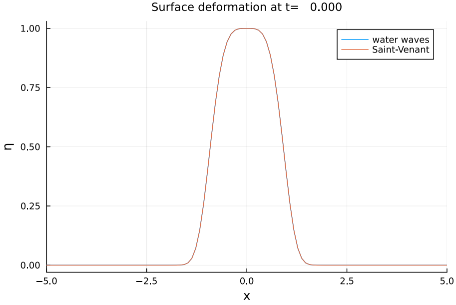
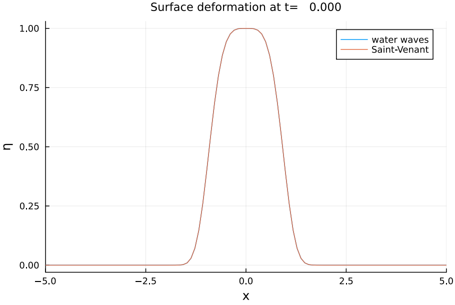
Outline

Failures of the shallow-water system



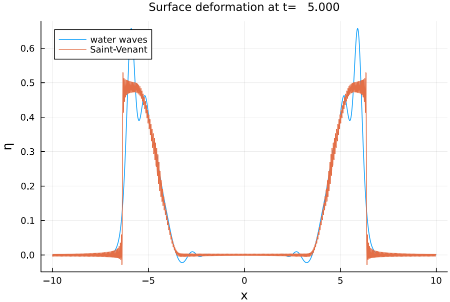
$\mu = 10^{-2},\ \epsilon = 1/4$



$\mu = 10^{-2},\ \epsilon = 10^{-2}$

Qn : Which situation can we improve by considering refined shallow-water models?

Failures of the shallow-water system



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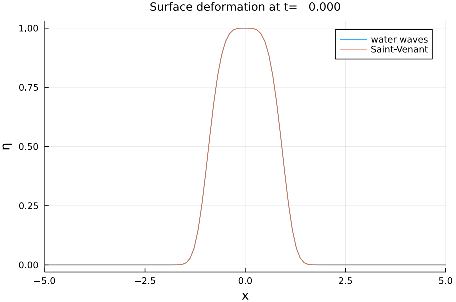

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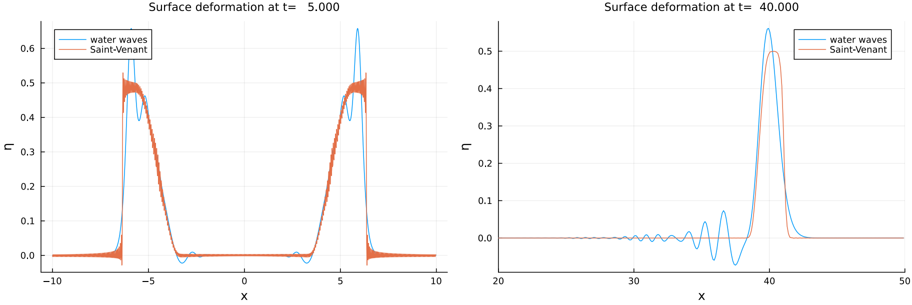
Failures of the shallow-water system



$\mu = 10^{-2}, \ \epsilon = 1/4$ $\qquad$ $\mu = 10^{-2}, \ \epsilon = 10^{-2}$

Qn : Which situation can we improve by considering refined <u>shallow-water</u> models?

Higher-order models

The **dispersion relation** associated with the water-waves system linearized about the rest state is (**Ex :** Prove it or check [Lannes (2013), §1.3.2] or [D. (2021), §ii.])

$$\omega^2(\mathbf{k}) = |\mathbf{k}|^2 \frac{\sinh(\sqrt{\mu}|\mathbf{k}|)}{(\sqrt{\mu}|\mathbf{k}|)\cosh(\sqrt{\mu}|\mathbf{k}|)}.$$

Some rational function approximations are

dispersion relation	model example
$\omega^2(\mathbf{k}) = \|\mathbf{k}\|^2$	shallow-water

Qn : Can you propose higher-order (rational function) approximations?

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Some rational function approximations are

dispersion relation	model example
$\omega^2(\mathbf{k}) = \lvert\mathbf{k}\rvert^2$	shallow-water
$\omega^2(\mathbf{k}) = \lvert\mathbf{k}\rvert^2(1 - \frac{1}{3}\mu\lvert\mathbf{k}\rvert^2)$	bad Boussinesq
$\omega^2(\mathbf{k}) = \lvert\mathbf{k}\rvert^2 \frac{1}{1+\frac{1}{3}\mu\lvert\mathbf{k}\rvert^2}$	Serre–Green–Naghdi
$\omega^2(\mathbf{k}) = \lvert\mathbf{k}\rvert^2 \frac{1+\frac{1}{6}\mu\lvert\mathbf{k}\rvert^2}{1+\frac{1}{2}\mu\lvert\mathbf{k}\rvert^2}$	Choi
$\omega^2(\mathbf{k}) = \lvert\mathbf{k}\rvert^2 \frac{1+\frac{1}{15}\mu\lvert\mathbf{k}\rvert^2}{1+\frac{2}{5}\mu\lvert\mathbf{k}\rvert^2}$	Isobe–Kakinuma

Qn : Can you propose higher-order (rational function) approximations?

Thoughts to go

- We have **rigorously justified** a shallow water model, following the roadmap

$$\text{consistency} + \text{stability} \Rightarrow \text{convergence.}$$

- We used tools for elliptic (consistency) and hyperbolic (stability) PDEs, **keeping track of the parameters.**
- Our strategy demands strong regularity assumptions, but offers **quantitative** results (such as convergence rates).
- A variety of models coexist, and none is perfect.

Bibliography

[Lannes. The water waves problem. Mathematical analysis and asymptotics, 2013.]

[Lannes. Modeling shallow water waves. *Nonlinearity*, 2020.]

[D. Many models for water waves. *Open Math Notes*, 2021.]

Next talks

III. Singular limits. Justification of the model used in the Tolosa simulation.
IV. and **V.** We discuss some aspects of shear velocities and variable density.