Modeling water waves and internal waves

Lecture 1 : Asymptotic modeling

Vincent Duchêne

Ravello School of Mathematical Physics

September 2026

Outline

Today

1. Why modeling matters
2. Modeling done wrong
3. Modeling done right
4. Summary and plan

Next days

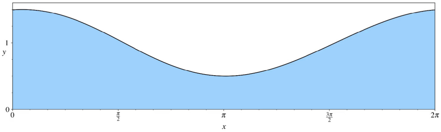
II. Shallow water models

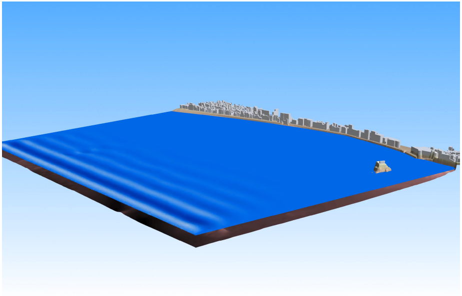
III. Singular limits

IV. Internal waves

V. Interfacial waves

Two recent achievements





https://ariquier.gitlab.io/

[Dormy & Lacave (2024)]

[Riquier & Dormy (2024)]

3 weeks (now would be 3 days) of simulation with 24 cores

https://tolosa-project.com/

Tolosa library (2021–...)

(team)

20 minutes of simulation on a MacBook Pro M2 (6 cores)

Qn : What are some differences between these two animations ?

Outline

Our master equations

The free-surface incompressible Euler equations read

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{U}) = 0 & \text{in } \Omega, \\ \rho\big(\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U}\big) + \nabla P = -g\rho \mathbf{e}_z & \text{in } \Omega, \\ \nabla \cdot \mathbf{U} = 0 & \text{in } \Omega, \end{cases} \tag{Euler}$$

where $\Omega(t) := \{(\mathbf{x}, z) \ : \ -H < z < \zeta(t, \mathbf{x})\}$ and

$$\begin{cases} \mathbf{U} \cdot \mathbf{e}_z = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = -H\}, \\ P = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = \zeta(t, \mathbf{x})\}, \\ \partial_t \zeta + \mathbf{U} \cdot \begin{pmatrix} \nabla_{\mathbf{x}} \zeta \\ -1 \end{pmatrix} = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = \zeta(t, \mathbf{x})\}, \end{cases} \tag{b.c.}$$

The initial-value problem requires to provide initial data

$$\zeta|_{t=0} = \zeta^0, \quad \mathbf{U}|_{t=0} = \mathbf{U}^0, \quad \rho|_{t=0} = \rho^0. \tag{i.d.}$$

Qn : What are some modelization assumptions ?

Well-posedness

Well-posedness [Fradin (2026)]

Let $(\zeta^0, \rho^0, \mathbf{U}^0)$ be suitable initial data in X^s a Sobolev-based Banach space with large enough but finite regularity ($s \geq s_0$).
Then there exists $T > 0$ and a unique $(\zeta, \rho, \mathbf{U}) \in C([0,T); X^s)$ local-in-time solution to (Euler)-(b.c.)-(i.d.).

Sketch of the proof.

$$\begin{cases} \partial_t \rho + \mathbf{U} \cdot \nabla \rho = 0 & \text{in } \Omega, \\ \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} + \frac{1}{\rho}\nabla P = -g\mathbf{e}_z & \text{in } \Omega, \\ \nabla \cdot \mathbf{U} = 0 & \text{in } \Omega, \end{cases} \qquad \text{(Euler)}$$

where P is determined by the elliptic equation

$$-\nabla \cdot (\tfrac{1}{\rho}\nabla P) = \nabla \cdot ((\mathbf{U} \cdot \nabla)\mathbf{U}) \qquad \text{(and b.c.)}$$

1. **Elliptic estimates** of the form $\|\nabla P\|_{H^s} \lesssim \|(\mathbf{U}, \rho)\|_{H^s}$ (see [Desjardins-Lannes-Saut (2021)])
2. (semi-)Lagrangian change of variables to deal with the **variable domain**.

The leading-order contribution of $\frac{1}{\rho}\nabla P + g\mathbf{e}_z$ together with $\mathbf{U} \cdot \begin{pmatrix} \nabla_x \zeta \\ -1 \end{pmatrix}$ is skew symmetric for a suitable L^2-based inner-product $\rightsquigarrow$ **energy estimates** (typical from wave propagation). $\square$

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Qn : Are we happy now?

Blessing and curse of the hydrostatic assumption

We set $\mathbf{U} =: (\mathbf{v}, w)$ and $\nabla =: (\nabla_{\mathbf{x}}, \partial_z)$ so our master equations read

$$\begin{cases} \partial_t \rho + \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho + w \partial_z \rho = 0 & \text{in } \Omega, \\ \rho\big(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}} + w\partial_z)\mathbf{v}\big) + \nabla_{\mathbf{x}} P = 0 & \text{in } \Omega, \\ \rho\big(\partial_t w + (\mathbf{v} \cdot \nabla_{\mathbf{x}} + w\partial_z) w\big) + \partial_z P = -g\rho & \text{in } \Omega, \\ \nabla_{\mathbf{x}} \cdot \mathbf{v} + \partial_z w = 0 & \text{in } \Omega, \end{cases} \tag{Euler}$$

where $\Omega(t) := \{(\mathbf{x}, z) \ : \ -H < z < \zeta(t, \mathbf{x})\}$ and

$$\begin{cases} w = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = -H\}, \\ P = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = \zeta(t, \mathbf{x})\}, \\ \partial_t \zeta + \mathbf{v} \cdot \nabla_{\mathbf{x}} \zeta - w = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = \zeta(t, \mathbf{x})\}. \end{cases} \tag{b.c.}$$

Let us neglect the vertical advection

$$\rho\big(\partial_t w + (\mathbf{v} \cdot \nabla_{\mathbf{x}} + w\partial_z) w\big) + \partial_z P = -g\rho \quad \rightsquigarrow \partial_z P_{\mathrm{h}} = -g\rho, \ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{v}$$

The system becomes...

Blessing and curse of the hydrostatic assumption

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The system becomes...

$$\begin{cases} \partial_t \rho_{\rm h} + \mathbf{v}_{\rm h}\cdot\nabla_{\mathbf{x}}\rho_{\rm h} + w_{\rm h}\partial_z\rho_{\rm h} = 0 & \text{in } \Omega, \\ \rho_{\rm h}\big(\partial_t \mathbf{v}_{\rm h} + (\mathbf{v}_{\rm h}\cdot\nabla_{\mathbf{x}} + w_{\rm h}\partial_z)\mathbf{v}_{\rm h}\big) + \nabla_{\mathbf{x}} P_{\rm h} = 0 & \text{in } \Omega, \\ \partial_t \zeta_{\rm h} + \mathbf{v}_{\rm h}|_{z=\zeta}\cdot\nabla_{\mathbf{x}}\zeta_{\rm h} - w_{\rm h}|_{z=\zeta} = 0 \end{cases} \qquad \text{(hydro)}$$

where

$$\begin{cases} w_{\rm h}(t,\mathbf{x},z) = -\int_{-H}^{z}(\nabla_{\mathbf{x}}\cdot\mathbf{v}_{\rm h})(t,\mathbf{x},z')\mathrm{d}z', \\ P_{\rm h}(t,\mathbf{x},z) = g\int_{z}^{\zeta(t,\mathbf{x})}\rho_{\rm h}(t,\mathbf{x},z')\mathrm{d}z'. \end{cases}$$

$\rightsquigarrow$ The equations are local in the variable $\mathbf{x}$!

Well-posedness?

Blessing and curse of the hydrostatic assumption

Let us neglect the vertical advection

$$\rho\big(\partial_t w + (\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)w\big) + \partial_z P = -g\rho \quad \rightsquigarrow \partial_z P_{\rm h} = -g\rho,\ \partial_z w = -\nabla_{\mathbf{x}}\cdot\mathbf{v}$$

The system becomes...

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Well-posedness?

The initial-value problem for the equations (hydro) in finite-regularity spaces is ill-posed in general, and open otherwise.

Qn : What went wrong?

Outline

Asymptotic modeling is the key

A suitable rescaling of the variables $(t, \mathbf{x}, z)$ and unknowns $(\rho, \mathbf{v}, w, \zeta)$ allows to identify crucial **dimensionless parameters**:

The aspect ratio (μ); Density variations (γ); The size of the waves (ϵ);

and to write **dimensionless equations**:

$$\begin{cases} \partial_t \tilde\rho + \epsilon \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde\rho + \epsilon w \partial_z \tilde\rho = 0 & \text{in } \Omega, \\ (1+\gamma\epsilon\tilde\rho)\big(\partial_t \mathbf{v} + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)\mathbf{v}\big) + \nabla_{\mathbf{x}} P = -\nabla_{\mathbf{x}}\zeta & \text{in } \Omega, \\ \mu(1+\gamma\epsilon\tilde\rho)\big(\partial_t w + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)w\big) + \partial_z P = -\gamma\tilde\rho & \text{in } \Omega, \\ \nabla_{\mathbf{x}} \cdot \mathbf{v} + \partial_z w = 0 & \text{in } \Omega, \end{cases} \tag{Euler}$$

where $\Omega(t) := \{(\mathbf{x}, z) \ : \ -1 < z < \epsilon\zeta(t,\mathbf{x})\}$ and

$$\begin{cases} w = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = -1\}, \\ P = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = \epsilon\zeta(t,\mathbf{x})\}, \\ \partial_t \zeta + \epsilon \mathbf{v}\cdot\nabla_{\mathbf{x}}\zeta - w = 0 & \text{on } \{(\mathbf{x}, z) \ : \ z = \epsilon\zeta(t,\mathbf{x})\}. \end{cases} \tag{b.c.}$$

Ex : Perform the non-dimensionalization

Asymptotic modeling is the key

$$\begin{cases} \partial_t\tilde\rho + \epsilon\mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde\rho + \epsilon w\partial_z\tilde\rho = 0 & \text{in } \Omega,\\ (1+\gamma\epsilon\tilde\rho)\big(\partial_t\mathbf{v} + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)\mathbf{v}\big) + \nabla_{\mathbf{x}}P = -\nabla_{\mathbf{x}}\zeta & \text{in } \Omega,\\ \mu(1+\gamma\epsilon\tilde\rho)\big(\partial_t w + \epsilon(\mathbf{v}\cdot\nabla_{\mathbf{x}} + w\partial_z)w\big) + \partial_z P = -\gamma\tilde\rho & \text{in } \Omega,\\ \nabla_{\mathbf{x}}\cdot\mathbf{v} + \partial_z w = 0 & \text{in } \Omega, \end{cases} \qquad \text{(Euler)}$$

where $\Omega(t) \coloneqq \{(\mathbf{x},z) \;:\; -1 < z < \epsilon\zeta(t,\mathbf{x})\}$ and

$$\begin{cases} w = 0 & \text{on } \{(\mathbf{x},z) \;:\; z = -1\},\\ P = 0 & \text{on } \{(\mathbf{x},z) \;:\; z = \epsilon\zeta(t,\mathbf{x})\},\\ \partial_t\zeta + \epsilon\mathbf{v}\cdot\nabla_{\mathbf{x}}\zeta - w = 0 & \text{on } \{(\mathbf{x},z) \;:\; z = \epsilon\zeta(t,\mathbf{x})\}. \end{cases} \qquad \text{(b.c.)}$$

Well-posedness [Fradin (2026)]

There exists $T > 0$ **such that** for all $(\zeta^0, \rho^0, \mathbf{v}^0, \sqrt{\mu}w^0)$ suitable initial data of size 1 in X^s ($s \geq s_0$) and such that $\big\|\partial_z\mathbf{v}^0 - \mu\nabla_x w^0\big\|_{H^s} \lesssim \sqrt{\mu}$, there is a unique $(\zeta, \rho, \mathbf{v}, w) \in C([0, T/(\epsilon+\gamma/\mu)]; X^s)$ local-in-time solution to (Euler)-(b.c.)-(i.d.).

Qn : What do we learn?

The limit $\mu \to 0$

From the previous result we can expect to **rigorously derive**[1] the hydrostatic free-surface Euler equations in the limit of small aspect ratio $\mu \to 0$ (a.k.a. **shallow water limit**), at least when $\gamma = 0$ (homogeneous flows) and the vorticity vanishes (potential flows).

For this we have two main strategies

1. uniform bounds + compactness $\Rightarrow$ (weak) convergence
2. consistency + stability $\Rightarrow$ (strong) convergence

Qn : What should we do?

[1] In the sense that solutions to the original (dimensionless) free-surface Euler eqns are well-defined on a non-vanishing time interval and converge in some sense to a solution to the hydrostatic equations.

Thoughts to go

- We have emphasized the importance of models when simulating water waves.
- We have cautioned against blind formal approaches for deriving models.
- After rescaling (**Ex :** do it!), a handful of parameters describe (fairly roughly) the physical situation we consider. Assumptions on the size of these parameters embed (fairly roughly) the modeling assumptions; this is **asymptotic modeling**.
- From a mathematical viewpoint, this leads us to investigate **PDEs with parameters**, and to keep track on the influence of these parameters in theoretical results.
- Remember the limit PDE is not your target; it is (often) a powerful and convenient way to characterize the asymptotic behavior of your physical system.

Next talks

II. Principles in action: we fully justify a (shallow-water) model.
III. Singular limits. Justification of the model used in the Tolosa simulation.
IV. and **V.** We discuss some aspects of shear velocities and variable density.