On WW2

Propagation of deep water waves

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Joint work with Benjamin Melinand (Paris Dauphine)

The model

$$\begin{cases} \partial_t\zeta - |D|\psi + \epsilon|D|(\zeta|D|\psi) + \epsilon\nabla\cdot(\zeta\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0, \end{cases} \tag{WW2}$$

where $|D| = (-\Delta_{\mathbf{x}})^{1/2}$, $\mathbf{x}\in\mathbb{R}^d$, $d\in\{1,2\}$.

- (WW2) is a model for water waves in infinite depth, assuming small steepness, $\epsilon \ll 1$.
- (WW2) enjoys a Hamiltonian structure. In particular, it preserves

$$\int \zeta \, d\mathbf{x}, \quad \int \zeta\nabla\psi \, d\mathbf{x},$$

$$\mathcal{H}_1(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + \psi|D|\psi + \epsilon\zeta\left(|\nabla\psi|^2 - (|D|\psi)^2\right) d\mathbf{x}.$$

- (WW2) belongs to a hierarchy of models [Craig&Sulem '93] based on the converging asymptotic expansion

$$\mathcal{H}(\zeta,\psi) \stackrel{\text{def}}{=} \mathcal{H}_1(\zeta,\psi) + \frac{1}{2}\int_{\mathbb{R}^d} \epsilon^2\psi\mathcal{G}_2[\zeta,\zeta]\psi + \epsilon^3\psi\mathcal{G}_3[\zeta,\zeta,\zeta]\psi + \dots$$

Numerical instabilities

Numerical integration of the systems in the hierarchy are easily and efficiently implemented using Fourier spectral methods (as done in e.g. [Guyenne&Nicholls '07-08])

> *In the computations [...] it was observed that spurious oscillations can develop in the wave profile, due to the onset of an instability related to the growth of numerical errors at high wavenumbers. [...] Similar high-wavenumber instabilities were observed by other authors [...] who used smoothing techniques to circumvent this difficulty.*

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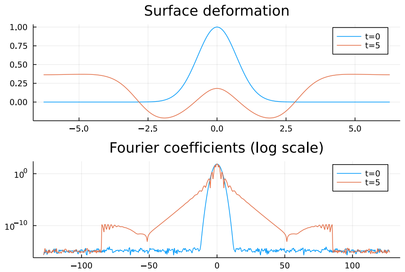
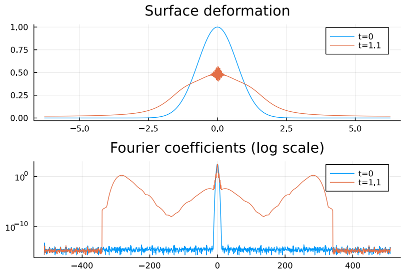


$\epsilon = 1/4, N = 2^9, L = 4\pi, \mathrm{dt} = 10^{-3}$



$\epsilon = 1/4, N = 2^{11}, L = 4\pi, \mathrm{dt} = 10^{-3}$

Proposed instability mechanism

[Ambrose,Bona&Nicholls '14] suggest that

$$\begin{cases} \partial_t\zeta - |D|\psi + \epsilon|D|(\zeta|D|\psi) + \epsilon\nabla\cdot(\zeta\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0, \end{cases} \tag{WW2}$$

[and also (WW3)] is ill-posed in Sobolev spaces, based on

- tailored numerical experiments;
- the toy model

$$\partial_t\psi + \frac{\epsilon}{2}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0. \tag{toy}$$

[Ambrose,Bona&Nicholls '14]

For all $s \in [0,3)$, the Cauchy problem associated with (toy) is ill-posed[a] in $H^s(\mathbb{T})$.

[a] there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth solutions to (toy) defined on $t \in [0, T_n)$ and such that $\left|\psi_n(0;\cdot)\right|_{H^s} \searrow 0$, $T_n \searrow 0$ as $n \to \infty$ and $\left|\psi_n(t,\cdot)\right|_{L^2} \to \infty$ as $t \nearrow T_n$.

Outline

Quasi-linearization

$$\begin{cases} \partial_t \zeta - |D|\psi + \epsilon |D|(\zeta |D|\psi) + \epsilon \nabla \cdot (\zeta \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0, \end{cases} \qquad \text{(WW2)}$$

Compensation Lemma [Saut&Xu '12]

Let $d \in \{1,2\}$, $t_0 > d/2$. For all $r \leq 1$ and $s \geq t_0 + r$,

$$\big||D|(f|D|g) + \nabla \cdot (f \nabla g)\big|_{H^s} \lesssim \big|\nabla f\big|_{H^s}\big|\nabla g\big|_{H^{s-r}}.$$

Proof ($d = 1$). Denote $a = |D|(f|D|g) + \partial_x(f\partial_x g)$. For $\xi \geq 0$,

$$\widehat{a}(\xi) = \int_{\mathbb{R}} \left(|\xi||\xi-\eta| - \xi(\xi-\eta)\right)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\,\mathrm{d}\eta = 2\int_\xi^\infty \xi(\eta-\xi)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\,\mathrm{d}\eta.$$

Since $|\xi| \leq |\eta|$ and $|\eta - \xi| \leq |\eta|$, one has for all $s \geq 0$ and $r' \geq 0$

$$\langle\xi\rangle^s|\widehat{a}(\xi)| \leq 2\int_{\mathbb{R}}\langle\eta\rangle^{s+r'}|\eta||\widehat{f}(\eta)|\langle\xi-\eta\rangle^{-r'}|\xi-\eta||\widehat{g}(\xi-\eta)|\,\mathrm{d}\eta.$$

By Young's inequality and since $\langle\cdot\rangle^{-t_0} \in L^2(\mathbb{R})$, with $r' = 0$,

$$\big||D|(f|D|g) + \nabla \cdot (f\nabla g)\big|_{H^s} \lesssim \big|\partial_x f\big|_{H^s}\big|\partial_x g\big|_{H^{t_0}}.$$

This shows the result, without restriction on r when $d = 1$. □

Quasi-linearization

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The principle part of the first equation is

$$\partial_t \dot\zeta - |D|\dot\psi + \epsilon |D|(\dot\zeta |D|\underline{\psi}) + \epsilon \nabla \cdot (\dot\zeta \nabla \underline{\psi}) = 0.$$

The principle part of the second equation is

$$\partial_t \dot\psi + \dot\zeta + \epsilon(\nabla \underline{\psi}) \cdot (\nabla \dot\psi) - \epsilon(|D|\underline{\psi})(|D|\dot\psi) = 0.$$

One recognizes *Alinhac's good unknown*: $\check\psi \overset{\text{def}}{=} \dot\psi - \epsilon(|D|\psi)\dot\zeta$:

$$\begin{cases} \partial_t \dot\zeta - |D|\check\psi + \epsilon \nabla \cdot (\dot\zeta \nabla \underline{\psi}) = 0, \\ \partial_t \check\psi + \mathfrak{a}[\underline{\zeta}, \underline{\psi}]\dot\zeta + \epsilon\left(\nabla \underline{\psi} \cdot \nabla \check\psi\right) = 0, \end{cases} \tag{Q-WW2}$$

with

$$\mathfrak{a}[\underline{\zeta}, \underline{\psi}]f \overset{\text{def}}{=} \underbrace{f - \epsilon(|D|\underline{\zeta})f}_{a(\underline{\zeta})f} - \epsilon^2 \underbrace{(|D|\underline{\psi})|D|\{(|D|\underline{\psi})f\}}_{☠}.$$

Quasi-linearization

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Toy model

We mimic

$$\begin{cases} \partial_t \dot{\zeta} - |D|\check{\psi} + \epsilon \nabla \cdot (\dot{\zeta} \nabla \underline{\psi}) = 0, \\ \partial_t \check{\psi} + \mathfrak{a}[\underline{\zeta}, \underline{\psi}]\dot{\zeta} + \epsilon \big(\nabla \underline{\psi} \cdot \nabla \check{\psi} \big) = 0, \end{cases} \tag{Q-WW2}$$

$$\mathfrak{a}[\underline{\zeta}, \underline{\psi}] f \stackrel{\mathrm{def}}{=} \underbrace{f - \epsilon(|D|\underline{\zeta}) f}_{a(\underline{\zeta}) f} - \epsilon^2 \underbrace{(|D|\underline{\psi})|D|\big\{(|D|\underline{\psi}) f\big\}}_{\text{☠}}.$$

with

$$\begin{cases} \partial_t \zeta - |D|\psi = 0, \\ \partial_t \psi + \big(1 - \alpha[\psi]|D|\big)\zeta = 0, \end{cases} \qquad \alpha[\psi] \stackrel{\mathrm{def}}{=} \epsilon^2 \int (|D|\psi)^2 \, \mathrm{d}\mathbf{x}. \tag{toy}$$

Toy model

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Ill-posedness in $H^\infty(\mathbb{T}^d)$ [D-Melinand]

For all $\epsilon > 0$, there exists $(\zeta_n, \psi_n)_{n\in\mathbb{N}}$ smooth solutions to (toy) defined on $[0, T_n)$ with

$$\forall s \in \mathbb{R}, \ \left|\zeta_n\big|_{t=0}\right|_{H^s(\mathbb{T}^d)} + \left|\psi_n\big|_{t=0}\right|_{H^s(\mathbb{T}^d)} \searrow 0 \quad \text{and} \quad T_n \searrow 0 \quad (n \to \infty),$$

and

$$\forall s' \in \mathbb{R}, \qquad \left|\psi^n(t,\cdot)\right|_{H^{s'}(\mathbb{T}^d)} \to \infty \quad (t \nearrow T_n).$$

Proof. We put $\zeta_n\big|_{t=0} = 0$ and $\psi_n\big|_{t=0} = b_n \cos(\mathbf{k}_0 \cdot \mathbf{x}) + c_n \cos(\mathbf{k}_n \cdot \mathbf{x})$ where

$$\mathbf{k}_0 \neq \mathbf{0}, \ |\mathbf{k}_n| \nearrow \infty, \ b_n = |\mathbf{k}_n|^{-1/4}, \ c_n = \exp(-|\mathbf{k}_n|^{1/4}).$$

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$$\mathbf{k}_0 \neq \mathbf{0},\ |\mathbf{k}_n| \nearrow \infty,\ b_n = |\mathbf{k}_n|^{-1/4},\ c_n = \exp(-|\mathbf{k}_n|^{1/4}).$$

By studying the system of ODEs on Fourier coefficients, we observe successively

- low-high instabilities: $\alpha[\psi]|\mathbf{k}_0| < 1$ but $\alpha[\psi_0]|\mathbf{k}_n| > 2 \ \Rightarrow c_n(t) \geq \frac{c_n}{8}e^{|\mathbf{k}_n|^{1/2}t}$,
- high-high instabilities: $c_n(t) \gtrsim |\mathbf{k}_n|^{-1} \Rightarrow \alpha[\psi_n]|\mathbf{k}_n| > 2 \Rightarrow c_n(t) \geq \frac{c_n}{8}e^{|\mathbf{k}_n|^{1/2}t}$,
- high-high blow-up: $c_n(t) \gtrsim 1 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\psi}^n_{\mathbf{k}_n} \geq \frac{1}{4}|\mathbf{k}_n|^{1/2}\widehat{\psi}^n_{\mathbf{k}_n}(t)^2 \Rightarrow$ blow-up.

□

Numerical validation

Numerical integration of (WW2) initial data

$$\zeta(t=0,x)=0 \quad \text{et} \quad \psi'(t=0,x)=\left(\sin(x)+\frac{\sin(Kx)}{K^2}\right)\exp(-|x|^2).$$

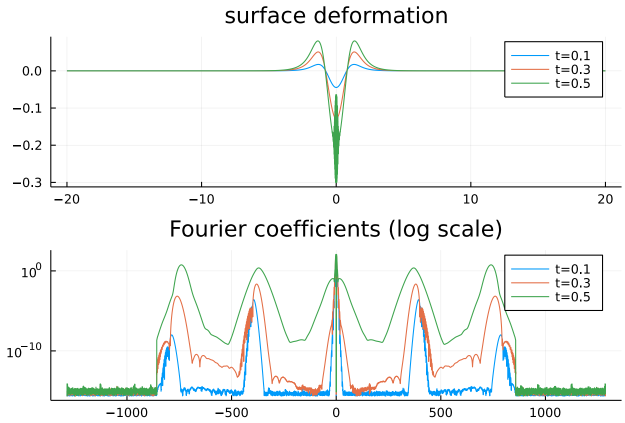

Figure: Time integration with $\epsilon = 1/5$, $K = 400$.

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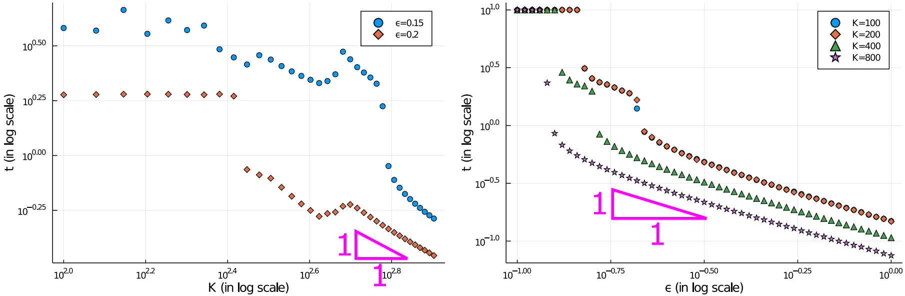

Figure: Blow-up time $T^\star$ depending on K and ϵ.

The toy model predicts $T^\star \propto (\epsilon K)^{-1}$ if $\epsilon^2 K \gg 1$.

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Quasi-linearization

$$\begin{cases} \partial_t\zeta - |D|\psi + \epsilon|D|((\mathsf{J}\zeta)|D|\psi) + \epsilon\nabla\cdot((\mathsf{J}\zeta)\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0. \end{cases} \tag{WW2}$$

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$$\begin{cases} \partial_t\dot\zeta - |D|\check\psi + \epsilon\nabla\cdot((\mathsf{J}\dot\zeta)\nabla\underline{\psi}) = 0, \\ \partial_t\check\psi + \mathfrak{a}_{\mathsf{J}}[\underline{\zeta},\underline{\psi}]\dot\zeta + \epsilon\left(\nabla\underline{\psi}\cdot\nabla\check\psi\right) = 0, \end{cases} \tag{Q-WW2}$$

with

$$\mathfrak{a}_{\mathsf{J}}[\underline{\zeta},\underline{\psi}]f \stackrel{\text{def}}{=} \underbrace{f - \epsilon(|D|\underline{\zeta})\mathsf{J}f}_{a(\underline{\zeta})\mathsf{J}f} - \epsilon^2\underbrace{(|D|\underline{\psi})\mathsf{J}|D|\{(|D|\underline{\psi})\mathsf{J}f\}}_{☺}.$$

The regularized sysem

By plugging $\mathsf{J} = J(D)$ self-adjoint,

$$\begin{cases} \partial_t \zeta - |D|\psi + \epsilon |D|((\mathsf{J}\zeta)|D|\psi) + \epsilon \nabla \cdot ((\mathsf{J}\zeta)\nabla\psi) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2}\mathsf{J}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0, \end{cases} \qquad \text{(RWW2)}$$

enjoys a canonical hamiltonian structure, with

$$\mathcal{H}_1^{\mathsf{J}}(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + \psi|D|\psi + \epsilon(\mathsf{J}\zeta)\left(|\nabla\psi|^2 - (|D|\psi)^2\right) \, \mathbf{dx}.$$

Well-posedness [D-Melinand]

Let $\mathsf{J} = J(D)$ with $J \lesssim \langle\cdot\rangle^{-1}$. Let $s \geq d/2 + 1/2$. For all $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d)$, there exists a unique maximal solution $(\zeta,\psi) \in \mathcal{C}((-T_\star, T^\star); H^s(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d))$ to (RWW2), $(\zeta,\psi)\,|_{t=0} = (\zeta_0,\psi_0)$. Moreover, if $J \lesssim \langle\cdot\rangle^{-m}$, $m > d/2 + 3/2$ and ϵ small enough, then $T_\star = T^\star = +\infty$.

Proof. Compensation lemma [Saut&Xu'12] + Duhamel formula.

For ϵ small enough, $\mathcal{H}_1^{\mathsf{J}}(\zeta,\psi) \approx \left|\zeta\right|_{L^2}^2 + \left||D|^{1/2}\psi\right|_{L^2}^2$ is preserved. □

The cost of regularizing

Consistency [D-Melinand]

If $\mathsf{J} = J_0(\delta D)$ with $J_0 \in L^\infty(\mathbb{R}^d)$ and $|\cdot|^{-\ell}(1-J_0) \in L^\infty(\mathbb{R}^d)$, then for all $\delta > 0$, $s > d/2$ and $(\zeta,\psi) \in \mathcal{C}([0,T]; H^{\max(s+\ell+1,s+2)}(\mathbb{R}^d) \times H^{\max(s+\ell+\frac{3}{2})}(\mathbb{R}^d))$ solution to (RWW2), (ζ,ψ) satisfies (WW) up to remainder terms

$$\big|R_0\big|_{H^s\times H^{s+\frac12}} \lesssim \epsilon^2 \Big(\big|\zeta\big|_{H^{s+2}} + \big||D|^{1/2}\psi\big|_{H^{s+1}}\Big),$$
$$\big|R_{\mathsf{J}}\big|_{H^s\times H^{s+\frac12}} \lesssim \epsilon\,\delta^\ell \left(\big|\zeta\big|_{H^{s+\ell+1}} + \big|\nabla\psi\big|_{H^{s+\ell+\frac12}}\right).$$

Proof.

For R_0, [Alvarez-Samaniego&Lannes'08].

For R_{J}, $\big|f - \mathsf{J}f\big|_{H^s} \le C_\ell \delta^\ell \big|f\big|_{H^s}$, with $C_\ell = \big||\cdot|^{-\ell}(1-J_0)\big|_{L^\infty}$. □

The cost of regularizing

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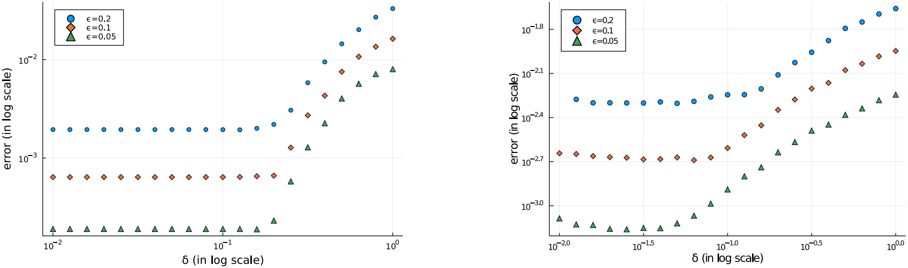
$$\big|R_0\big|_{H^s \times H^{s+\frac{1}{2}}} \lesssim \epsilon^2 \Big(\big|\zeta\big|_{H^{s+2}} + \big||D|^{1/2}\psi\big|_{H^{s+1}} \Big),$$

$$\big|R_{\mathsf{J}}\big|_{H^s \times H^{s+\frac{1}{2}}} \lesssim \epsilon\, \delta^\ell \Big(\big|\zeta\big|_{H^{s+\ell+1}} + \big|\nabla\psi\big|_{H^{s+\ell+\frac{1}{2}}} \Big).$$



Error between (RWW2) and (WW): smooth i.d. (left) and non-smooth i.d. (right).

The gain of regularizing

Large time well-posedness [D-Melinand]

Let $J_0 = J_0(|D|)$ with $\langle\cdot\rangle^{-1} J_0 \in L^\infty$, $\langle\cdot\rangle \nabla J \in L^\infty$. Let $s > d/2+2$, $s \in \mathbb{N}$, $C > 1$ and $M > 0$. There exists $T_0 > 0$ such that for all $\epsilon > 0$, for all $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$ with

$$0 < \epsilon M_0 \stackrel{\text{def}}{=} \epsilon\left(\left|\zeta_0\right|_{H^s} + \left||D|^{1/2}\psi_0\right|_{H^s}\right) \leq M,$$

and for all $\delta \geq \epsilon M_0$, one has: for all $J = J_0(\delta D)$, there exists a unique $(\zeta, \psi) \in \mathcal{C}([0, T_0/(\epsilon M_0)]; H^s \times H^{s+\frac{1}{2}})$ solution to (RWW2), $(\zeta, \psi)\,|_{t=0} = (\zeta_0, \psi_0)$, with

$$\sup_{t\in[-T_0/(\epsilon M_0), T_0/(\epsilon M_0)]} \left(\left|\zeta(t,\cdot)\right|^2_{H^s} + \left||D|^{1/2}\psi(t,\cdot)\right|^2_{H^s}\right) \leq C\left(\left|\zeta_0\right|^2_{H^s} + \left||D|^{1/2}\psi_0\right|^2_{H^s}\right).$$

Proof.

If $\epsilon \gtrsim 1$, Duhamel formula $\Rightarrow T_0 \approx \delta$.

If $\epsilon \ll 1$, energy method $\Rightarrow T_0 \approx \min(\{1, \delta/\epsilon\})$.

The gain of regularizing

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Let $J_0 = J_0(|D|)$ with $\langle\cdot\rangle^{-1} J_0 \in L^\infty$, $\langle\cdot\rangle \nabla J \in L^\infty$. Let $s > d/2 + 2$, $s \in \mathbb{N}$, $C > 1$ and $M > 0$. There exists $T_0 > 0$ such that for all $\epsilon > 0$, for all $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$ with

$$0 < \epsilon M_0 \stackrel{\text{def}}{=} \epsilon\left(\left|\zeta_0\right|_{H^s} + \left||D|^{1/2}\psi_0\right|_{H^s}\right) \leq M,$$

and for all $\delta \geq \epsilon M_0$, one has: for all $J = J_0(\delta D)$, there exists a unique $(\zeta, \psi) \in \mathcal{C}([0, T_0/(\epsilon M_0)]; H^s \times H^{s+\frac{1}{2}})$ solution to (RWW2), $(\zeta, \psi)\,|_{t=0} = (\zeta_0, \psi_0)$, with

$$\sup_{t \in [-T_0/(\epsilon M_0), T_0/(\epsilon M_0)]} \left(\left|\zeta(t,\cdot)\right|_{H^s}^2 + \left||D|^{1/2}\psi(t,\cdot)\right|_{H^s}^2\right) \leq C\left(\left|\zeta_0\right|_{H^s}^2 + \left||D|^{1/2}\psi_0\right|_{H^s}^2\right).$$

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Energy method

$$\begin{cases} \partial_t\zeta - |D|\psi + \epsilon|D|(\mathsf{J}\zeta)|D|\psi) + \epsilon\nabla\cdot((\mathsf{J}\zeta)\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}\mathsf{J}\left(|\nabla\psi|^2 - (|D|\psi)^2\right) = 0. \end{cases} \qquad \text{(RWW2)}$$

The principle part of the first equation is

$$\partial_t\dot\zeta - |D|\dot\psi + \epsilon|D|((\mathsf{J}\dot\zeta)|D|\underline{\psi}) + \epsilon\nabla\cdot((\mathsf{J}\dot\zeta)\nabla\underline{\psi}) = 0.$$

The principle part of the second equation is

$$\partial_t\dot\psi + \dot\zeta + \epsilon(\nabla\underline{\psi})\cdot\mathsf{J}(\nabla\dot\psi) - \epsilon(|D|\underline{\psi})\mathsf{J}(|D|\dot\psi) = 0.$$

One recognizes *Alinhac's good unknown*: $\check\psi \stackrel{\text{def}}{=} \dot\psi - \epsilon(|D|\psi)(\mathsf{J}\dot\zeta)$:

$$\begin{cases} \partial_t\dot\zeta - |D|\check\psi + \epsilon\nabla\cdot((\mathsf{J}\dot\zeta)\nabla\underline{\psi}) = 0, \\ \partial_t\check\psi + \mathfrak{a}_{\mathsf{J}}[\underline{\zeta},\underline{\psi}]\dot\zeta + \epsilon\left(\nabla\underline{\psi}\cdot\mathsf{J}\nabla\check\psi\right) = 0, \end{cases} \qquad \text{(Q-RWW2)}$$

with

$$\mathfrak{a}_{\mathsf{J}}[\underline{\zeta},\underline{\psi}]f \stackrel{\text{def}}{=} f - \epsilon(|D|\underline{\zeta})\mathsf{J}f - \epsilon^2(|D|\underline{\psi})\mathsf{J}^2|D|\{(|D|\underline{\psi})f\}.$$

and one has

$$\min(\epsilon,\epsilon^2\delta^{-1}) \ll 1 \quad \Longrightarrow \quad \left(\mathfrak{a}_{\mathsf{J}}[\underline{\zeta},\underline{\psi}]f, f\right)_{L^2} \geq \tfrac{1}{2}\big|f\big|_{L^2}^2.$$

Numerical validation

In numerical simulations, we observe a dichotomy:

- If $\delta > \delta_{\text{crit.}}$, then large time stability.
- If $\delta < \delta_{\text{crit.}}$, then rapid blow-up.

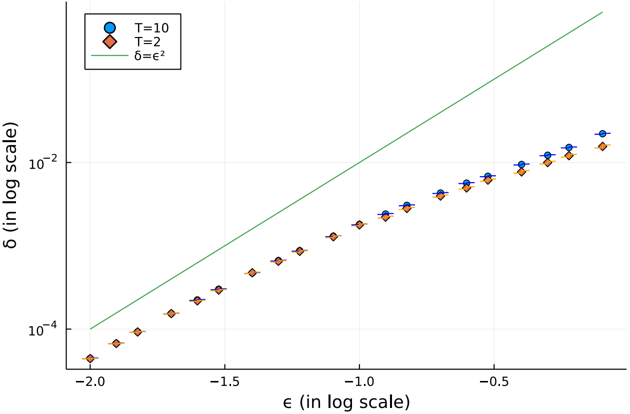

Figure: Critical value $\delta_{\text{crit.}}$, depending on ϵ.

Conclusion and perspectives

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We exhibited the instability mechanism induced by (WW2), and proposed a “rectification” which (for δ well-chosen)

- does not harm the precision (in the sense of consistency) of the model;
- restores large time well-posedness (and hence convergence);
- is costless form the point of view of numerical discretization.

Perspectives

- Results are proved in deep but finite depth, not in shallow water.
- We have not proved ill-posedness.
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Thank you for your attention