Asymptotic models for internal waves in Oceanography

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École Normale Supérieure de Paris

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Internal waves in ocean

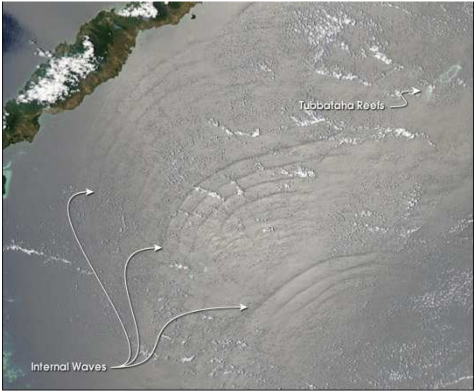

Figure: Sulu Sea. April 8, 2003[1]

▸ The large picture

[1] Credits: NASA's Earth Observatory (Picture of the Day July 1, 2003) http://earthobservatory.nasa.gov/IOTD/view.php?id=3586

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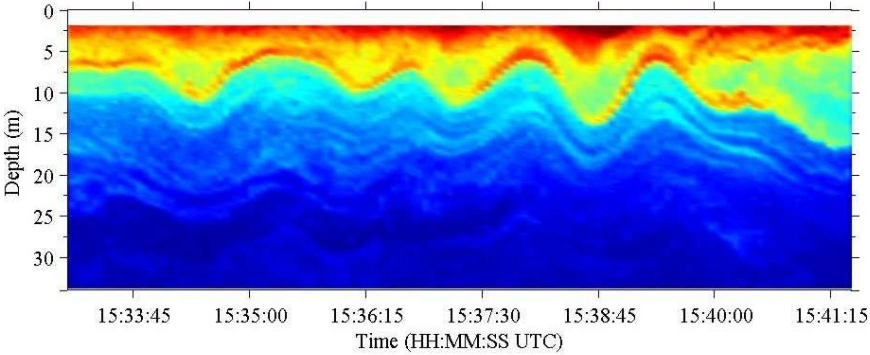
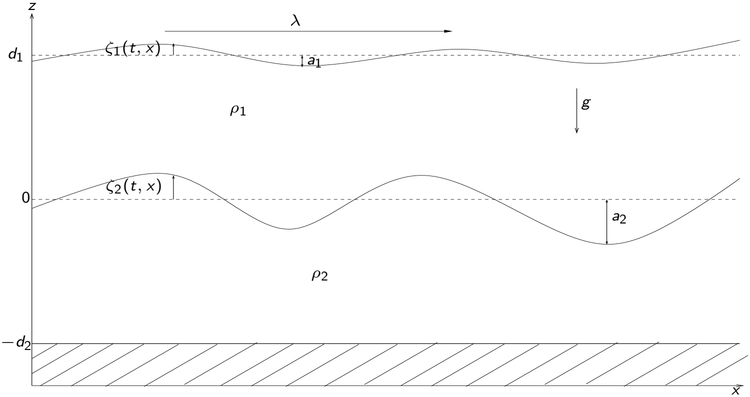


Figure: St. Lawrence Estuary[1]

[1]Credits: St. Lawrence Estuary Internal Wave Experiment (SLEIWEX) http://myweb.dal.ca/kelley/SLEIWEX/index.php

Two layers of immiscible, homogeneous, ideal, incompressible fluids



$$\epsilon_1 \equiv \frac{a_1}{d_1}, \quad \epsilon_2 \equiv \frac{a_2}{d_1}, \quad \mu \equiv \frac{d_1^2}{\lambda^2}, \quad \gamma \equiv \frac{\rho_1}{\rho_2}, \quad \delta \equiv \frac{d_1}{d_2}$$

$$\epsilon_1 = \epsilon_2 = \mu \equiv \varepsilon \ll 1, \quad \gamma \in (0,1), \quad 0 < \delta_{\min} \leq \delta \leq \delta_{\max} < +\infty$$

Outline

Hypotheses on the fluids

The assumptions

The fluid is irrotational

The fluid is homogeneous, incompressible

The fluid is inviscid

The fluid particles do not cross the bottom

The fluid particles do not cross the surface

The particles of the two fluids do not cross the interface.

The equations

$$\mathbf{v}_i = \nabla_{x,z}\phi_i \quad (i = 1, 2)$$

$$\Delta_{x,z}\phi_i = 0$$

$$\partial_t\phi_i + \frac{1}{2}|\nabla_{x,z}\phi_i|^2 = -\frac{P}{\rho_i} - gz$$

$$\partial_z\phi_2 = 0 \quad \text{on } \Gamma_b$$

$$\partial_t\zeta_1 = \sqrt{1 + |\partial_x\zeta_1|^2}\partial_n\phi_1 \quad \text{on } \Gamma_1$$

$$\partial_t\zeta_2 = \sqrt{1 + |\partial_x\zeta_2|^2}\partial_n\phi_1 = \sqrt{1 + |\partial_x\zeta_2|^2}\partial_n\phi_2 \quad \text{on } \Gamma_2.$$

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The equations

$$\begin{aligned}
\mathbf{v}_i &= \nabla_{x,z}\phi_i \quad (i = 1, 2)\\
\Delta_{x,z}\phi_i &= 0\\
\partial_t\phi_i + \tfrac{1}{2}|\nabla_{x,z}\phi_i|^2 &= -\frac{P}{\rho_i} - gz\\
\partial_z\phi_2 &= 0 \quad \text{on } \Gamma_b\\
\partial_t\zeta_1 &= \sqrt{1 + |\partial_x\zeta_1|^2}\partial_n\phi_1 \quad \text{on } \Gamma_1\\
\partial_t\zeta_2 &= \sqrt{1 + |\partial_x\zeta_2|^2}\partial_n\phi_1\\
&= \sqrt{1 + |\partial_x\zeta_2|^2}\partial_n\phi_2 \quad \text{on } \Gamma_2.
\end{aligned}$$

Additional assumptions

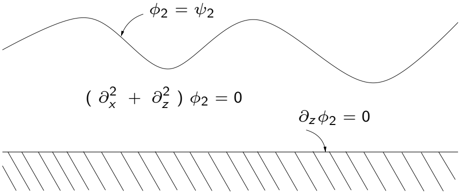
The fluid is at rest at infinity

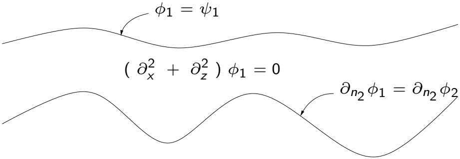
The pressure P is constant at the surface, and continuous at the interface

There is no surface tension

Dirichlet-Neumann operators

The equations can be reduced to evolution equations located on the surface and on the interface thanks to the following operators





$$\begin{cases} \Delta_{x,z}\phi_2 = 0 & \text{in } \Omega_2, \\ \phi_2 = \psi_2 & \text{on } \Gamma_2 \equiv \{z = \zeta_2\}, \\ \partial_z \phi_2 = 0 & \text{on } \Gamma_b \equiv \{z = -d_2\}, \end{cases}$$

$$\downarrow$$

$$\phi_2$$

$$\begin{cases} \Delta_{x,z}\phi_1 = 0 & \text{in } \Omega_1, \\ \phi_1 = \psi_1 & \text{on } \Gamma_1 \equiv \{z = d_1 + \zeta_1\}, \\ \partial_{n_2}\phi_1 = \partial_{n_2}\phi_2 & \text{on } \Gamma_2. \end{cases}$$

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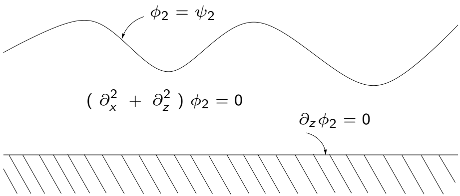

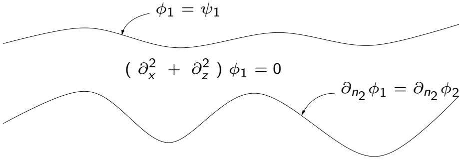

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Dirichlet-Neumann operators

The equations can be reduced to evolution equations located on the surface and on the interface thanks to the following operators

Definition (Dirichlet-Neumann operators)

The following operators are well-defined:

$$G_2\psi_2 \equiv \sqrt{1+|\partial_x\zeta_2|^2}\partial_n\phi_{2|\text{interface}},$$

$$G_1(\psi_1,\psi_2) \equiv \sqrt{1+|\partial_x\zeta_1|^2}\partial_n\phi_{1|\text{surface}},$$

$$H(\psi_1,\psi_2) \equiv \partial_x\Big(\phi_{1|z=\zeta_2}\Big).$$

Therefore, the system is entirely defined by

$$\zeta_1 \quad ; \quad \zeta_2 \quad ; \quad \psi_1 \equiv \phi_{1|\text{surface}} \quad ; \quad \psi_2 \equiv \phi_{2|\text{interface}}.$$

The full Euler system

Thanks to the the previous definitions, and after an adapted change of variables (obtained through the study of the linearized system), one obtains

The dimensionless full Euler system

$$(\Sigma)\quad \begin{cases} \partial_t\zeta_1 \; - \; \frac{1}{\varepsilon}G_1(\psi_1,\psi_2) = 0, \\ \partial_t\zeta_2 \; - \; \frac{1}{\varepsilon}G_2\psi_2 = 0, \\ \partial_t\partial_x\psi_1 \; + \; \partial_x\zeta_1 + \frac{\varepsilon}{2}\partial_x(|\partial_x\psi_1|^2) - \varepsilon^2\partial_x\mathcal{N}_1 = 0, \\ \partial_t(\partial_x\psi_2 - \gamma H(\psi_1,\psi_2)) \; + \; (1-\gamma)\partial_x\zeta_2 \; + \; \frac{\varepsilon}{2}\partial_x(|\partial_x\psi_2|^2 - \gamma|H(\psi_1,\psi_2)|^2) \\ \qquad\qquad - \varepsilon^2\partial_x\mathcal{N}_2 \; = \; 0, \end{cases}$$

Solutions of this system are exact solutions of our problem. We construct then asymptotic models, and therefore look for approximate solutions.

State of the art

- The one-layer problem in the long wave regime
  - Justification of asymptotic Boussinesq model and KdV approximation: **[Craig 1985]**, **[Schneider, Wayne 2000]**, **[BenYoussef, Colin 2000]**, **[Bona, Colin, Lannes 2005]**, **[Alvarez-Samaniego, Lannes 2008]**
- The two-layer problem, with a rigid lid
  - KdV equations for internal waves **[Keulegan, 1953]**, **[Long, 1956]**
  - Boussinesq-type models **[Miyata 1985]**, **[Choi, Camassa 1996]**
  - Hamiltonian formulation **[Lvov, Tabak 2004]**, **[Craig, Guyenne, Kalisch 2005]**
  - Consistency of Boussinesq-type models **[Bona, Lannes, Saut 2008]** ($d = 1$ or 2, with topography)
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Asymptotic expansion of the operators

Proposition

Let $s > 1$, $\zeta_1,\ \zeta_2,\ \psi_1,\ \psi_2\ \in\ H^{s+t}(\mathbb{R})$. Then one has

$$\Big| G_2\psi_2\ +\ \varepsilon\partial_x(h_2\partial_x\psi_2)\ +\ \varepsilon^2\frac{1}{3\delta^3}\partial_x^3\partial_x\psi_2)\Big|_{H^s}\ \leq\ \varepsilon^3 C$$

$$\Big| G_1(\psi_1,\psi_2)\ +\ \varepsilon\partial_x(h_1\partial_x\psi_1 + h_2\partial_x\psi_2)$$

$$+\ \varepsilon^2\partial_x^3\Big(\frac{1}{3}\partial_x\psi_1 + \big(\frac{1}{3\delta^3}+\frac{1}{2\delta}\big)\partial_x\psi_2\Big)\Big|_{H^s}\ \leq\ \varepsilon^3 C,$$

$$\Big| H(\psi_1,\psi_2)\ -\ \partial_x\psi_1\ -\ \varepsilon\partial_x^2\Big(\frac{1}{2}\partial_x\psi_1 + \frac{1}{\delta}\partial_x\psi_2\Big)\Big|_{H^s}\ \leq\ \varepsilon^2 C,$$

Notations: $h_1 = 1 + \varepsilon\zeta_1 - \varepsilon\zeta_2$ and $h_2 = \frac{1}{\delta} + \varepsilon\zeta_2$.

The Boussinesq/Boussinesq models

A Boussinesq/Boussinesq model

$$\begin{cases} \partial_t\zeta_1 + \partial_x(h_1\partial_x\psi_1) + \partial_x(h_2\partial_x\psi_2) = -\varepsilon\left(\frac{1}{3}\partial_x^4\psi_1 + (\frac{1}{3\delta^3} + \frac{1}{2\delta})\partial_x^4\psi_2\right), \\ \partial_t\zeta_2 + \partial_x(h_2\partial_x\psi_2) = -\varepsilon\frac{1}{3\delta^3}\partial_x^4\psi_2, \\ \partial_t\partial_x\psi_1 + \partial_x\zeta_1 + \frac{\varepsilon}{2}\partial_x\left(|\partial_x\psi_1|^2\right) = 0, \\ \partial_t\partial_x\psi_2 + (1-\gamma)\partial_x\zeta_2 + \gamma\partial_x\zeta_1 + \frac{\varepsilon}{2}\partial_x\left(|\partial_x\psi_2|^2\right) \\ \qquad = \varepsilon\partial_t\partial_x^2\left(\frac{\gamma}{\delta}\partial_x\psi_2 + \frac{\gamma}{2}\partial_x\psi_1\right), \end{cases}$$

$$\hookrightarrow \quad \partial_t U + A_0\partial_x U + \varepsilon\big(A_1(U)\partial_x U + B\partial_x^2\partial_t U + C\partial_x^3 U \big) = 0$$

with $U = (\zeta_1, \zeta_2, \partial_x\psi_1, \partial_x\psi_2)$.

Proposition (consistency)

The full Euler system is consistent with the Boussinesq/Boussinesq model, with precision $\mathcal{O}(\varepsilon^2)$.

The Boussinesq/Boussinesq models

A Boussinesq/Boussinesq model

$$(\mathcal{M}_B)\quad \partial_t U + A_0 \partial_x U + \varepsilon \left(A_1(U)\partial_x U + B\partial_x^2 \partial_t U + C\partial_x^3 U \right) = 0.$$

Proposition (consistency)

The full Euler system is consistent with the Boussinesq/Boussinesq model, with precision $\mathcal{O}(\varepsilon^2)$.

Let U be a strong solution of the full Euler system (Σ), uniformly bounded in a sufficiently high Sobolev norm. Then U satisfies the Boussinesq/Boussinesq model, up to some residuals, bounded (in H^s norm) by $\epsilon^2 C_0$.

Open questions:

- Well-posedness of any Boussinesq/Boussinesq system?
- Convergence of its solutions towards solutions of the full Euler system?

Symmetrization

The system can be written under the compact form

$$\partial_t U \; + \; A_0 \partial_x U \; + \; \varepsilon \; \big(\; A_1(U) \partial_x U \; + \; B \partial_x^2 \partial_t U \; + \; C \partial_x^3 U \; \big) \; = \; 0.$$

Multiply by adapted $\mathcal{S} \; \equiv \; \mathcal{S}_0 \; + \; \varepsilon \mathcal{S}_1(U) \; - \; \varepsilon \mathcal{S}_2 \partial_x^2$, and withdraw $O(\varepsilon^2)$ terms. One obtains a perfectly symmetric model of the form:

The symmetric Boussinesq/Boussinesq model

$$(\mathcal{S}_B) \quad \Big(S_0 + \varepsilon \big(S_1(U) - S_2 \partial_x^2 \big) \Big) \partial_t U + \Big(\Sigma_0 + \varepsilon \big(\Sigma_1(U) - \Sigma_2 \partial_x^2 \big) \Big) \partial_x U = 0,$$

with the following properties:

- Matrices S_0, S_2, Σ_0, $\Sigma_2 \in \mathcal{M}_4(\mathbb{R})$ are symmetric.
- $S_1(\cdot)$ and $\Sigma_1(\cdot)$ are linear mappings, with values in $\mathcal{M}_4(\mathbb{R})$, and for all $U \in \mathbb{R}^4$, $S_1(U)$ and $\Sigma_1(U)$ are symmetric.
- S_0 et S_2 are definite positive.

Consistency

The full Euler system is consistent with the symmetric Boussinesq/Boussinesq model $(\mathcal{S}_B)$, with precision $\mathcal{O}(\varepsilon^2)$.

Well posedness

The symmetric system is well-posed in H^{s+1} $(s > 3/2)$ over times of order $\mathcal{O}(1/\varepsilon)$. Moreover, one has the estimate

$$\Big(\big|U(t)\big|^2_{H^s} + \varepsilon\big|U(t)\big|^2_{H^{s+1}}\Big)^{1/2} = \big|U(t)\big|_{H^{s+1}_\varepsilon} \leq C_0 \frac{\big|U^0\big|_{H^{s+1}_\varepsilon}}{1 - C_0 t\varepsilon\big|U^0\big|_{H^{s+1}_\varepsilon}} \ .$$

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Convergence

The difference between any solution U of the full Euler system (Σ), and the solution U_B of the symmetric Boussinesq/Boussinesq model $(\mathcal{S}_B)$ with same initial data, satisfies

$$\forall t \in [0, T/\varepsilon], \qquad \big|U - U_B\big|_{L^\infty([0,t];H^{s+1}_\varepsilon)} \ \leq \ \varepsilon^2 t C_1.$$

▸ Numerical simulation

The WKB expansion

We seek an approximate solution of system $(\mathcal{S}_B)$:

$$\Big(S_0 + \varepsilon\big(S_1(U) - S_2\partial_x^2\big)\Big)\partial_t U + \Big(\Sigma_0 + \varepsilon\big(\Sigma_1(U) - \Sigma_2\partial_x^2\big)\Big)\partial_x U = 0$$

of the form $U_{\rm app}(t,x) \ \equiv \ U_0(\varepsilon t, t, x) \ + \ \varepsilon U_1(\varepsilon t, t, x).$

At order $\mathcal{O}(1)$: $\quad (S_0\partial_t + \Sigma_0\partial_x)U_0 \ = \ 0.$

There exists a basis $\mathbf{e}_i \in \mathbb{R}^4$ $(i = 1..4)$, diagonalizing S_0 and Σ_0:

$\Longrightarrow U_0 = \sum_{i=1}^{4} u_i\mathbf{e}_i$, with $\quad u_i(\tau, t, x) \ = \ u_i(\tau, x - c_i t).$

At order $\mathcal{O}(\varepsilon)$:

$$S_0\partial_\tau U_0 + \Sigma_1(U_0)\partial_x U_0 + S_1(U_0)\partial_t U_0$$
$$-\Sigma_2\partial_x^3 U_0 - S_2\partial_x^2\partial_t U_0 = -(S_0\partial_t + \Sigma_0\partial_x)U_1.$$

We split the equation in

$$\partial_\tau u_i \ + \ \lambda_i u_i \partial_{x_i} u_i \ + \ \mu_i \partial_{x_i}^3 u_i \ = \ 0,$$

$$(\partial_t + c_i\partial_x)\mathbf{e}_i \cdot U_1 \ + \sum_{(j,k)\neq(i,i)} \alpha_{ijk} u_k \partial_x u_j \ + \ \sum_{j\neq i} \beta_{ij}\partial_x^3 u_j \ = \ 0.$$

The KdV Approximation

Definition

Let U be a solution of the full Euler system (Σ). We define then the KdV approximation as $U_{KdV} = \sum_{i=1}^{4} u_i \mathbf{e}_i$, with u_i solution of

$$\text{(KdV)} \qquad \begin{cases} \partial_t u_i \ + \ c_i \partial_x u_i \ + \ \varepsilon \lambda_i u_i \partial_x u_i \ + \ \varepsilon \mu_i \partial_x^3 u_i \ = \ 0, \\ u_{i|t=0} \ = \ u_i^0, \end{cases}$$

Well-posedness

There exists a unique strong solution $U_0(\tau, t, x)$, uniformly bounded in $L^\infty([0, T] \times \mathbb{R}; H^{s+2}_\varepsilon)$.
Then, there exists an explicit residual $U_1 \in C^1([0, T) \times \mathbb{R}; H^s)$.

Secular growth of the residual

$$\forall(\tau, t) \in [0, T] \times \mathbb{R}, \quad \big|U_1(\tau, t, \cdot)\big|_{H^s} \leq C_0\sqrt{t}.$$

Moreover, if $(1 + x^2)U_0 \in H^{s+1}$, then one has the uniform estimate

$$\big|U_1(\tau, t, \cdot)\big|_{H^s} \leq C_0,$$

Consistency

$U_0(\varepsilon t, t, x) \ + \varepsilon U_1(\varepsilon t, t, x)$ satisfies the symmetric Boussinesq/Boussinesq model $(\mathcal{S}_B)$, with precision $\mathcal{O}(\varepsilon^{3/2})$ (and $\mathcal{O}(\varepsilon^2)$ if $(1 + x^2)U^0 \in H^{s+1}$).
$\Longrightarrow$ convergence towards the solution of $(\mathcal{S}_B)$.

Convergence towards solutions of the full Euler system

The difference between any solution U of the full Euler system (Σ), and $U_{\text{KdV}} \equiv \sum_{i=1}^{4} u_i \mathbf{e}_i$, with u_i solutions of (KdV), satisfies

$$\big| U - U_{\text{KdV}} \big|_{L^\infty([0,t];H_\varepsilon^{s+1})} \leq \varepsilon \sqrt{t} C_0,$$

Moreover, if $(1+x^2)U\big|_{t=0} \in H^{s+4}$, then one has the uniform estimate

$$\big| U - U_{\text{KdV}} \big|_{L^\infty([0,T/\varepsilon];H_\varepsilon^{s+1})} \leq \varepsilon C_0.$$

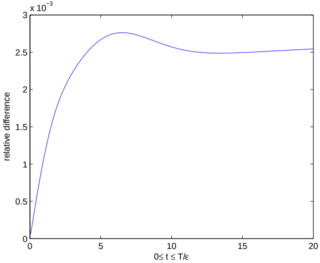
Why localization in space is relevant ?

▸ Propagation of a soliton

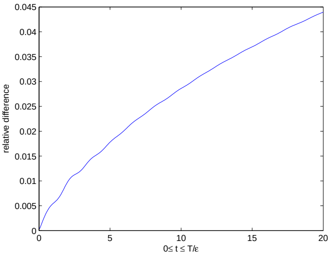
▸ Splitting of a bell curve

Why localization in space is relevant ?

Propagation of a soliton



Splitting of a bell curve



Comparison between free surface and rigid lid configurations.

Big density difference: $\gamma = 1/4$

Small density difference: $\gamma = 0.9$

Fridtjof Nansen, 1898

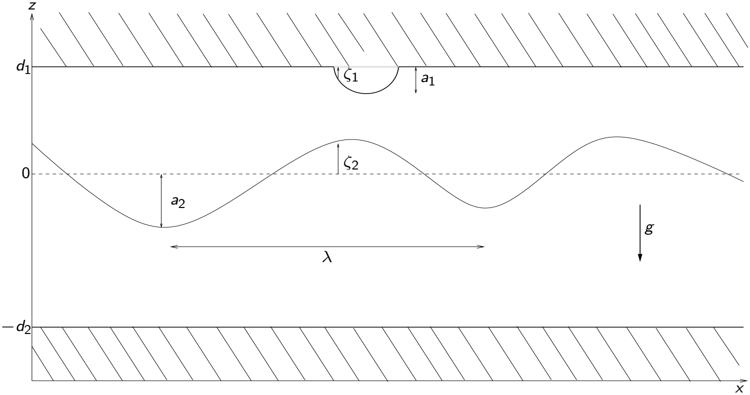
This peculiar phenomenon [...] manifests itself in the form of larger or smaller ripples or waves stretching across the wake, the one behind the other, arising sometimes as far forward as almost midships. **When caught in dead water, "Fram" appeared to be held back**, *as if by some mysterious force, and she did not always answer the helm. In calm weather, with a light cargo, "Fram" was* **capable of 6 to 7 knots. When in dead water she was unable to make 1.5 knots.** *We made loops in our course, turned sometimes right around, tried all sorts of antics to get clear of it, but to very little purpose.*

Vilhelm Bjerknes, 1900

In my reply to Prof. Nansen I remarked that in the case of a layer of fresh water resting on the top of salt water, a ship will not only produce the ordinary visible waves at the boundary between the water and the air, but will also generate invisible waves in the salt-water freshwater boundary below; **I suggested that the great resistance experienced by the ship was due to the work done in generating these [internal] waves**. *(...) In December 1899 I consequently suggested a pupil of mine, Dr. V.* **Walfrid Ekman** *(...) that he should do some simple preliminary experiments.*

▸ video of the experiments of Vasseur-Mercier-Dauxois, 2008

Rigid lid, but not flat : $\zeta_1(t,x) \equiv \zeta_1(x - c_s t)$



$$\epsilon_1 \equiv \frac{a_1}{d_1}, \quad \epsilon_2 \equiv \frac{a_2}{d_1}, \quad \mu \equiv \frac{d_1^2}{\lambda^2}, \quad \gamma \equiv \frac{\rho_1}{\rho_2}, \quad \delta \equiv \frac{d_1}{d_2}, \quad \mathrm{Fr} = \frac{c_s}{c_0}.$$

$$\epsilon_2 = \mu = \epsilon_1/\epsilon_2 \equiv \varepsilon \ll 1, \quad \gamma \in (0,1), \quad \delta \in [\delta_{\min}, \delta_{\max}], \quad \mathrm{Fr} \in (0, +\infty).$$

Wave (making) resistance suffered by the body

Definition (Wave resistance)

$$R_W \equiv \int_{\Gamma_{\text{ship}}} P\,(-\mathbf{e}_x \cdot \mathbf{n})\; dS = -\int_{\mathbb{R}} P_{|d_1+\zeta_1} \partial_x \zeta_1 \; dx.$$

where Γ_{ship} is the exterior domain of the ship, P is the pressure, $\mathbf{e}_x$ is the horizontal unit vector and $\mathbf{n}$ the normal unit vector exterior to the ship.

As a solution of the Bernoulli equation, the pressure P satisfies

$$\frac{P(x,z)}{\rho_1} = -\partial_t \phi_1(x,z) - \frac{1}{2}|\nabla_{x,z}\phi_1(x,z)|^2 - gz.$$

Using the previous change of variables, we define the dimensionless wave resistance coefficient C_W. In our regime ($\epsilon_2 = \mu = \epsilon_1/\epsilon_2 \equiv \varepsilon \ll 1$), the first order approximation is

$$C_W = \int_{\mathbb{R}} \zeta_1 \partial_x \zeta_2 \; dx + \mathcal{O}(\varepsilon).$$

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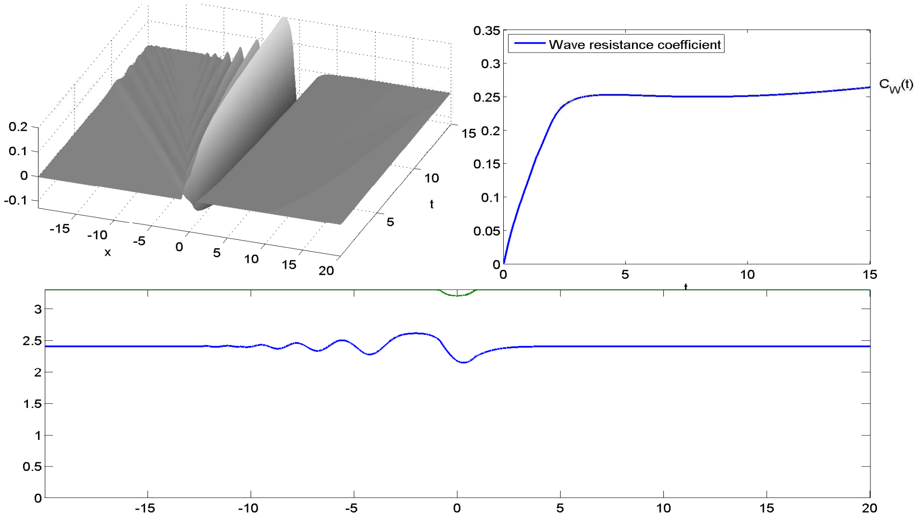
Numerical simulation



The dimensionless full Euler system

The governing equations can be deduced from the full Euler system in the free surface case:

The full Euler system when $\zeta_1(t,x) \equiv \zeta_1(x - \mathrm{Fr}\, t)$

$$\begin{cases} \varepsilon\partial_t\zeta_1 - \frac{1}{\varepsilon}G_1(\psi_1,\psi_2) = 0, \\ \partial_t\zeta_2 - \frac{1}{\varepsilon}G_2\psi_2 = 0, \\ \cancel{\partial_t\partial_x\psi_1 + \partial_x\zeta_1 + \frac{\varepsilon}{2}\partial_x(|\partial_x\psi_1|^2) - \varepsilon^2\partial_x\mathcal{N}_1 = 0,} \\ \partial_t(\partial_x\psi_2 - \gamma H(\psi_1,\psi_2)) + (1-\gamma)\partial_x\zeta_2 + \frac{\varepsilon}{2}\partial_x(|\partial_x\psi_2|^2 - \gamma|H(\psi_1,\psi_2)|^2) \\ \qquad - \varepsilon^2\partial_x\mathcal{N}_2 = 0, \end{cases}$$

The dimensionless full Euler system

The governing equations can be deduced from the full Euler system in the free surface case:

The dimensionless full Euler system with rigid lid

$$(\tilde{\Sigma}) \quad \begin{cases} -\varepsilon \, \mathrm{Fr} \, \partial_x \zeta_1 \; - \; \frac{1}{\varepsilon} G_1(\psi_1, \psi_2) \; = \; 0, \\ \partial_t \zeta_2 \; - \; \frac{1}{\varepsilon} G_2 \psi_2 \; = \; 0, \\ \partial_t \Big(\partial_x \psi_2 - \gamma H(\psi_1, \psi_2) \Big) + (\gamma + \delta) \partial_x \zeta_2 + \frac{\varepsilon}{2} \partial_x \Big(|\partial_x \psi_2|^2 - \gamma |H(\psi_1, \psi_2)|^2 \Big) \\ \qquad = \; \varepsilon^2 \partial_x \mathcal{N}_2, \end{cases}$$

Using that ζ_1 is a fixed parameter, the system reduces to two evolution equations for (ζ_2, v), with v the shear velocity defined by

$$v \; \equiv \; \partial_x \Big((\phi_2 - \gamma \phi_1)_{|z=\varepsilon\zeta_2} \Big) \; = \; \partial_x \psi_2 - \gamma H(\psi_1, \psi_2).$$

The Boussinesq-type models

Plug the asymptotic expansion of the operators G_1, G_2, H into the full Euler system $(\tilde{\Sigma})$, and withdraw $\mathcal{O}(\varepsilon^2)$ terms.

Boussinesq/Boussinesq model

$$\begin{cases} \partial_t \zeta_2 \ + \ \frac{1}{\delta+\gamma}\partial_x v + \varepsilon \frac{\delta^2-\gamma}{(\gamma+\delta)^2}\partial_x(\zeta_2 v) + \varepsilon \frac{1+\gamma\delta}{3\delta(\delta+\gamma)^2}\partial_x^3 v \ = \ -\varepsilon \frac{\mathrm{Fr}\,\gamma}{\delta+\gamma}\partial_x \zeta_1, \\ \partial_t v \ + \ (1-\gamma)\partial_x \zeta_2 \ + \ \frac{\varepsilon}{2}\frac{\delta^2-\gamma}{(\gamma+\delta)^2}\partial_x\left(|v|^2\right) \ = \ 0. \end{cases}$$

The Boussinesq/Boussinesq model system can be written as

$$\partial_t U \ + \ A_0 \partial_x U \ + \ \varepsilon A_1(U)\partial_x U \ - \ \varepsilon A_2 \partial_x^3 U \ = \ \varepsilon b_0(x - \mathrm{Fr}\, t),$$

The Boussinesq-type models

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When we multiply by adapted symmetrizer $S(U)\equiv S_0+\varepsilon S_1(U)-\varepsilon S_2\partial_x^2$, one gets

$$\left(\ S_0+\varepsilon S_1(U)-\varepsilon S_2\partial_x^2\ \right)\partial_t U \ + \ \left(\ \Sigma_0+\varepsilon\Sigma_1(U)-\varepsilon\Sigma_2\ \right)\partial_x U \ = \ \varepsilon b(x-\mathrm{Fr}\,t),$$

- The symmetric Boussinesq model is consistent at order $\mathcal{O}(\varepsilon^2)$,
- The symmetric Boussinesq model is well-posed, + energy estimate,
- $\Longrightarrow$ The solutions of the Boussinesq model converge towards solutions of the full Euler system $(\tilde{\Sigma})$, at order $\mathcal{O}(\varepsilon^2 t)$, for $t\in[0,T/\varepsilon]$.

The Boussinesq-type models

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The KdV approximation

Definition

Let $U = (\zeta_2, v)$ be a solution of the full Euler system $(\tilde{\Sigma})$. We define then the KdV approximation as $U_{KdV} = \big(\ \eta_+ + \eta_-\ ,\ (\gamma+\delta)(\eta_+ - \eta_-)\ \big)$, with $\eta_\pm$ solution of

$$\begin{cases} \partial_t \eta_\pm \pm \partial_x \eta_\pm \pm \varepsilon \frac{3}{2} \frac{\delta^2 - \gamma}{\gamma+\delta} \eta_\pm \partial_x \eta_\pm \pm \varepsilon \frac{1}{6} \frac{1+\gamma\delta}{\delta(\gamma+\delta)} \partial_x^3 \eta_\pm = -\varepsilon \,\mathrm{Fr}\, \gamma \partial_x \zeta_1, \\ \eta_{\pm|t=0} = \eta_\pm^0, \end{cases}$$

Convergence theorem

The difference between any solution U of the full Euler system $(\tilde{\Sigma})$, and U_{KdV} is bounded by

$$\big| U - U_{\mathrm{KdV}} \big|_{L^\infty([0,t];H^{s+1}_\varepsilon)} \leq \varepsilon \sqrt{t}\, C_0.$$

Moreover, if $(1+x^2)U\big|_{t=0} \in H^{s+4}$, then one has the uniform estimate

$$\big| U - U_{\mathrm{KdV}} \big|_{L^\infty([0,T/\varepsilon];H^{s+1}_\varepsilon)} \leq \varepsilon C_0.$$

A simple application

Lemma

Let u be the solution of

$$\partial_t u \;+\; c\partial_x u \;+\; \varepsilon\lambda u\partial_x u \;+\; \varepsilon\nu\partial_x^3 u \;=\; \varepsilon\partial_x f(x - c_0 t),$$

with $u_{|t=0} \;=\; \varepsilon u^0 \in H^{s+3}$, $s > 3/2$.

There exists $T(|\frac{1}{c-c_0}|)$ and $C(|\frac{1}{c-c_0}|) > 0$ such that

$$\left|\; u \;\right|_{L^\infty([0,T/\varepsilon];H^s)} \;\leq\; C\varepsilon.$$

The transport equation $\partial_t v \;+\; c\partial_x v \;=\; \varepsilon\partial_x f(x - c_0 t)$ leads to

$$v \;=\; \varepsilon u^0(x - ct) \;+\; \frac{\varepsilon}{c_0 - c}\big(\, f(x - c_0 t) - f(x - ct)\,\big)\,.$$

The result is obtained by comparison with this function.

As a consequence, the dead-water phenomenon will always be small if the velocity of the body is away from the critical velocity ($|\,\mathrm{Fr}\,| \;=\; 1$).

A simple application

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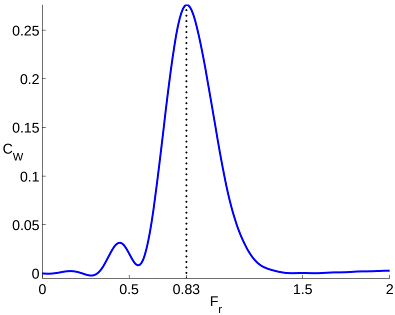

Wave resistance coefficient C_W at time $T = 10$, depending on the Froude number Fr, with $\delta = 1$ and 2 ($\gamma = 0.9$, $\varepsilon = 0.1$).

For more details:

Coupled models (Boussinesq, shallow water, Green-Naghdi): *Asymptotic shallow water models for internal waves in a two-fluid system with a free surface*, SIAM J. Math. Anal., **42** (2010)

KdV approximation: *Boussinesq/Boussinesq systems for internal waves with a free surface, and the KdV approximation*, to appear in Math. Model. Numer. Anal. (M2AN)

Dead-water phenomenon: *Asymptotic models for the generation of internal waves by a moving ship, and the dead-water phenomenon*, preprint Arxiv:1012.5892.

Thank you for your attention !