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Solitary Wave Solutions to a Class of Modified Green–Naghdi Systems

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Abstract. We provide the existence and asymptotic description of solitary wave solutions to a class of modified Green–Naghdi systems, modeling the propagation of long surface or internal waves. This class was recently proposed by Duchêne et al. (Stud Appl Math 137:356–415, 2016) in order to improve the frequency dispersion of the original Green–Naghdi system while maintaining the same precision. The solitary waves are constructed from the solutions of a constrained minimization problem. The main difficulties stem from the fact that the functional at stake involves low order non-local operators, intertwining multiplications and convolutions through Fourier multipliers.

1. Introduction

1.1. Motivation

In this work, we study solitary traveling waves for a class of long-wave models for the propagation of surface and internal waves. Starting with the serendipitous discovery and experimental investigation by John Scott Russell, the study of solitary waves at the surface of a thin layer of water in a canal has a rich history [20]. In particular, it is well-known that the most widely used nonlinear and dispersive models for the propagation of surface gravity waves, such as the Korteweg–de Vries equation or the Boussinesq and Green–Naghdi systems, admit explicit families of solitary waves [7,14,31,41,42]. These equations can be derived as asymptotic models for the so-called water waves system, describing the motion of a two-dimensional layer of ideal, incompressible, homogeneous, irrotational fluid with a free surface and a flat impermeable bottom; we let the reader refer to [32] and references therein for a detailed account of the rigorous justification of these models. Among them, the Green–Naghdi model is the most precise, in the sense that it does not assume that the surface deformation is small. However, the validity of all these models relies on the hypothesis that the depth of the layer is thin compared with the horizontal wavelength of the flow and, as expected, the models do not describe accurately the behavior of medium or short waves. In order to tackle this issue, one of the authors has recently proposed in [23] a new family of models:

$$\begin{cases} \partial_t \zeta + \partial_x w \;=\; 0, \\ \partial_t \left(h^{-1} w + \mathcal{Q}^{\mathsf{F}}[h](h^{-1}w) \right) \;+\; g\partial_x \zeta \;+\; \frac{1}{2}\partial_x \Big((h^{-1}w)^2 \Big) \;=\; \partial_x \big(\mathcal{R}^{\mathsf{F}}[h, h^{-1}w] \big), \end{cases} \tag{1.1}$$

where

$$\mathcal{Q}^{\mathsf{F}}[h]\overline{u} \overset{\text{def}}{=} -\frac{1}{3} h^{-1}\partial_x \mathsf{F}\big\{ h^3 \partial_x \mathsf{F}\{\overline{u}\}\big\},$$
$$\mathcal{R}^{\mathsf{F}}[h, \overline{u}] \overset{\text{def}}{=} \frac{1}{3}\overline{u} h^{-1}\partial_x \mathsf{F}\big\{ h^3 \partial_x \mathsf{F}\{\overline{u}\}\big\} + \frac{1}{2}\big(h\partial_x \mathsf{F}\{\overline{u}\}\big)^2.$$

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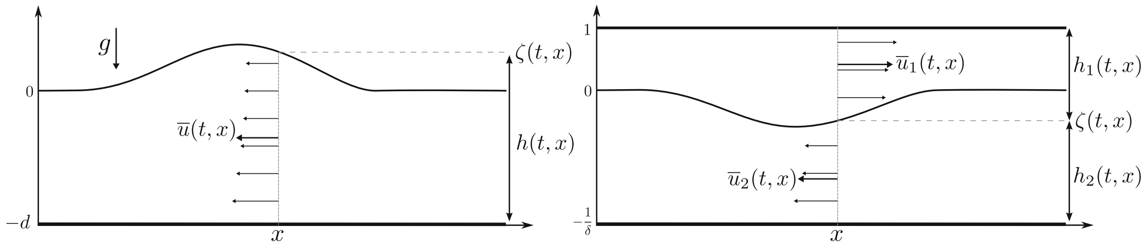

Fig. 1. Sketch of the domain and notations in the one-layer and bilayer situations

Here, ζ is the surface deformation, $h = d + \zeta$ the total depth (where d is the depth of the layer at rest), $\overline{u}$ the layer-averaged horizontal velocity, $w = h\overline{u}$ the horizontal momentum and g the gravitational acceleration; see Fig. 1. Finally, $\mathsf{F} \overset{\text{def}}{=} \mathsf{F}(D)$ is a Fourier multiplier, *i.e.*

$$\widehat{\mathsf{F}\{\varphi\}}(k) = \mathsf{F}(k)\widehat{\varphi}(k).$$

The original Green–Naghdi model is recovered when setting $\mathsf{F}(k) \equiv 1$. Any other choice satisfying $\mathsf{F}(k) = 1 + \mathcal{O}(k^2)$ enjoys the same precision (in the sense of consistency) in the shallow-water regime and the specific choice of $\mathsf{F}(k) = \sqrt{\frac{3}{d|k|\tanh(d|k|)} - \frac{3}{d^2|k|^2}}$ allows to obtain a model whose linearization around constant states fits exactly with the one of the water waves system. Hence system (1.1) with the aforementioned choice of Fourier multipliers participates to the recent effort in providing long wave models with the full dispersion property; see [1,11,28,30,39]. However, notice that contrarily to the so-called Boussinesq-Whitham equations, the validity of (1.1) does not rely on any small-amplitude assumption. The systems also preserve the Hamiltonian structure of the Green–Naghdi model, which turns out to play a key role in our analysis since the existence of solitary waves will be deduced from a variational principle.

The study of [23] is not restricted to surface propagation, but is rather dedicated to the propagation of internal waves at the interface between two immiscible fluids, confined above and below by rigid, impermeable and flat boundaries. Such a configuration appears naturally as a model for the ocean, as salinity and temperature may induce sharp density stratification, so that internal solitary waves are observed in many places [27,29,40]. Due to the weak density contrast, the observed solitary waves typically have much larger amplitude than their surface counterpart, hence the bilayer extension of the Green–Naghdi system introduced by [17,35,38], often called Miyata–Choi–Camassa model, is a very natural choice. It however suffers from strong Kelvin–Helmholtz instabilities—in fact stronger than the ones of the bilayer extension of the water waves system for large frequencies—and the work in [23] was motivated by taming these instabilities. The modified bilayer system reads

$$\begin{cases} \partial_t\zeta \ + \ \partial_x w \ = \ 0, \\ \partial_t\left(\dfrac{h_1+\gamma h_2}{h_1 h_2}w + \mathcal{Q}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w\right) \ + \ (\gamma+\delta)\partial_x\zeta \ + \ \dfrac{1}{2}\partial_x\Big(\dfrac{h_1^2-\gamma h_2^2}{(h_1h_2)^2}w^2\Big) = \partial_x\big(\mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,w]\big), \end{cases} \tag{1.2}$$

where we denote $h_1 = 1-\zeta$, $h_2 = \delta^{-1}+\zeta$, $\mathcal{Q}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w \overset{\text{def}}{=} \mathcal{Q}^{\mathsf{F}}_2[h_2](h_2^{-1}w) + \gamma\mathcal{Q}^{\mathsf{F}}_1[h_1](h_1^{-1}w)$ and $\mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,w] \overset{\text{def}}{=} \mathcal{R}^{\mathsf{F}}_2[h_2,h_2^{-1}w] - \gamma\mathcal{R}^{\mathsf{F}}_1[h_1,h_1^{-1}w]$, with

$$\mathcal{Q}^{\mathsf{F}}_i[h_i]\overline{u}_i \overset{\text{def}}{=} -\frac{1}{3}h_i^{-1}\partial_x\mathsf{F}_i\big\{h_i^3\partial_x\mathsf{F}_i\{\overline{u}_i\}\big\},$$
$$\mathcal{R}^{\mathsf{F}}_i[h_i,\overline{u}_i] \overset{\text{def}}{=} \frac{1}{3}\overline{u}_i h_i^{-1}\partial_x\mathsf{F}_i\big\{h_i^3\partial_x\mathsf{F}_i\{\overline{u}_i\}\big\} + \frac{1}{2}\big(h_i\partial_x\mathsf{F}_i\{\overline{u}_i\}\big)^2.$$

Here, ζ represents the deformation of the interface, h_1 (resp. h_2) is the depth of the upper (resp. lower) layer, $\overline{u}_1$ (resp. $\overline{u}_2$) is the layer-averaged horizontal velocity of the upper (resp. lower) layer and finally

$w = h_1 h_2(\overline{u}_2 - \gamma\overline{u}_1)/(h_1 + \gamma h_2)$ is the shear momentum. In this formulation we have used dimensionless variables, so that the depth at rest of the upper layer is scaled to 1, whereas the one of the lower layer is δ^{-1}, in which δ is the ratio of the depth at rest of the upper layer to the depth at rest of the lower layer (see Fig. 1). Similarly, γ is the ratio of the upper layer over the lower layer densities. As a consequence of our scaling, the celerity of infinitesimally small and long waves is $c_0 = 1$. Once again, F_i $(i = 1, 2)$ are Fourier multipliers. The choice $\mathsf{F}_i^{\mathrm{id}}(k) \equiv 1$ yields the Miyata–Choi–Camassa model while the system with

$$\mathsf{F}_i^{\mathrm{imp}}(k) = \sqrt{\frac{3}{\delta_i^{-1}|k|\tanh(\delta_i^{-1}|k|)} - \frac{3}{\delta_i^{-2}|k|^2}},$$

with convention $\delta_1 = 1, \delta_2 = \delta$, has the full dispersion property, namely its linearization around constant states fits exactly the one of the bilayer extension of the water waves system. Note that compared to equations (7)–(9) in [23] we have scaled the variables so that the shallowness parameter μ and amplitude parameter ϵ do not appear in the equations. This is for notational convenience since the parameters do not play a direct role in our results. On the other hand, we only expect the above model to be relevant for describing water waves in the regime $\mu \ll 1$ and the solutions that we construct in the end are found in the long-wave regime $\epsilon, \mu \ll 1$.

In the following, we study solitary waves for the bilayer system (1.2), noting that setting $\gamma = 0$ immediately yields the corresponding result for the one-layer situation, namely system (1.1). Our results are valid for a large class of parameters γ, δ and Fourier multipliers $\mathsf{F}_1, \mathsf{F}_2$, described hereafter. Our results are twofold:

(i) We prove the existence of a family of solitary wave solutions for system (1.2);
(ii) We provide an asymptotic description for this family in the long-wave regime.

These solitary waves are constructed from the Euler–Lagrange equation associated with a constrained minimization problem, as made possible by the Hamiltonian structure of system (1.2). There are however several difficulties compared with standard works in the literature following a similar strategy (see e.g. [2] and references therein). Our functional cannot be written as the sum of the linear dispersive contribution and the nonlinear pointwise contribution: Fourier multipliers and nonlinearities are entangled. What is more, the operators involved are typically of low order (F is a smoothing operator). In order to deal with this situation, we follow a strategy based on penalization and concentration-compactness used in a number of recent papers on the water waves problem (see e.g. [8, 9, 25] and references therein) and in particular, in a recent work by one of the authors on nonlocal model equations with weak dispersion [24]. Thus we show that the strategy therein may be favorably applied to bidirectional systems of equations in addition to unidirectional scalar equations such as the Whitham equation. Roughly speaking, the strategy in [24] is the following. The minimization problem is first solved in periodic domains using a penalization argument do deal with the fact that the energy functional is not coercive. This allows to construct a *special minimizing sequence* for the real line problem by letting the period tend to infinity, which is essential to rule-out the dichotomy scenario in Lions' concentration-compactness principle. The long-wave description follows from precise asymptotic estimates and standard properties of the limiting (Korteweg–de Vries) model. We follow closely this strategy, yet some additional difficulties arise in our situation. Firstly, the necessary estimates require more involved calculations, and in particular rely on (well-known) composition and product estimates in Sobolev spaces. Moreover, contrarily to the setting in [24], the operators involved in our functionals are of low but positive order ($1 - \theta \in (0, 1]$). As a consequence, a specific care is necessary to show the existence of a minimizer at the critical regularity, and we employ an approach based on [3]. However, the situation is simpler when the Fourier multipliers F_i have sufficiently high order ($-\theta \in (-1/2, 0]$) as we can in fact avoid the penalization argument and consider the minimization problem on the real line directly, since any minimizing sequence is then also a special minimizing sequence. In particular, this is the case for the original Miyata–Choi–Camassa model (and of course also the Green–Naghdi system). Finally, in order to ensure the smoothness of the constructed solitary waves, we need elliptic estimates on the Euler–Lagrange equation, which turns out to require tools from paradifferential calculus in the bilayer situation.

Our existence proof unfortunately gives no information about stability, since our variational formulation does not involve conserved functionals; see the discussion in Sect. 1.2. If sufficiently strong surface tension is included in the model, we expect that a different variational formulation could be used which also yields a conditional stability result (see [8,9,25]). A similar situation appears e.g. in the study of Boussinesq systems [15,16].

1.2. The Minimization Problem

We now set up the minimization problem which allows to obtain solitary waves of system (1.2). We seek traveling waves of (1.2), namely solutions of the form (abusing notation) $\zeta(t,x)=\zeta(x-ct)$, $w(t,x)=w(x-ct)$; from which we deduce

$$-c\partial_x\zeta+\partial_x w=0;\quad -c\partial_x\big(\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w\big)+(\gamma+\delta)\partial_x\zeta+\frac{1}{2}\partial_x\left(\frac{h_1^2-\gamma h_2^2}{h_1^2h_2^2}w^2\right)-\partial_x\big(\mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,w]\big)=0,$$

where we denote

$$\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w\overset{\text{def}}{=}\mathcal{A}^{\mathsf{F}}_2[h_2](h_2^{-1}w)+\gamma\mathcal{A}^{\mathsf{F}}_1[h_1](h_1^{-1}w),\quad \mathcal{A}^{\mathsf{F}}_i[h_i]\overline{u}_i\overset{\text{def}}{=}\overline{u}_i+\mathcal{Q}^{\mathsf{F}}_i[h_i]\overline{u}_i.$$

Integrating these equations and using the assumption (since we restrict ourselves to solitary waves) $\lim_{|x|\to\infty}\zeta(x)=\lim_{|x|\to\infty}w(x)=0$ yields the system of equations

$$\begin{cases}-c\zeta+w=0,\\[2ex] -c\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w+(\gamma+\delta)\zeta+\dfrac{1}{2}\dfrac{h_1^2-\gamma h_2^2}{(h_1h_2)^2}w^2\ -\ \mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,w]=0.\end{cases}\tag{1.3}$$

We now observe that system (1.2) enjoys a Hamiltonian structure. Indeed, define the functional

$$\mathcal{H}(\zeta,w)\overset{\text{def}}{=}\frac{1}{2}\int_{-\infty}^{\infty}(\gamma+\delta)\zeta^2+w\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w\,\mathrm{d}x.$$

Under reasonable assumptions on $\mathsf{F}_1,\mathsf{F}_2$, and for sufficiently regular ζ, $\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]$ defines a well-defined, symmetric, positive definite operator [23]. We may thus introduce the variable

$$v\overset{\text{def}}{=}\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w,\tag{1.4}$$

and write

$$\mathcal{H}(\zeta,v)\overset{\text{def}}{=}\frac{1}{2}\int_{-\infty}^{\infty}(\gamma+\delta)\zeta^2+v(\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta])^{-1}v\,\mathrm{d}x.$$

It is now straightforward to check that (1.2) can be written in terms of functional derivatives of $\mathcal{H}$:

$$\partial_t\zeta=-\partial_x\left(\mathrm{d}_v\mathcal{H}\right);\quad \partial_t v=-\partial_x\left(\mathrm{d}_\zeta\mathcal{H}\right).\tag{1.5}$$

Hence traveling waves are critical points of the functional $\mathcal{H}-c\mathcal{I}$ where

$$\mathcal{H}(\zeta,v)=\frac{1}{2}\int_{-\infty}^{\infty}(\gamma+\delta)\zeta^2+v\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]^{-1}v\,\mathrm{d}x\quad\text{ and }\quad\mathcal{I}(\zeta,v)\overset{\text{def}}{=}\int_{-\infty}^{\infty}\zeta v\,\mathrm{d}x.$$

However, as noticed (for the Green–Naghdi system) in [33], critical points are neither minimizers nor maximizers. We shall obtain solutions to (1.3) from a *constrained* minimization problem depending solely on the variable ζ. Notice that for each fixed c and ζ, the functional $v\mapsto\mathcal{H}(\zeta,v)-c\mathcal{I}(\zeta,v)$ has a unique critical point, $v_{c,\zeta}=c\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]\zeta$. Substituting $v_{c,\zeta}$ into $\mathcal{H}(\zeta,v)-c\mathcal{I}(\zeta,v)$, we obtain

$$\mathcal{H}(\zeta,v_{c,\zeta})-c\mathcal{I}(\zeta,v_{c,\zeta})=\frac{\gamma+\delta}{2}\big\|\zeta\big\|_{L^2}^2-\frac{c^2}{2}\mathcal{I}(\zeta,\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]\zeta).$$

Observe now that (ζ, v) is a critical point of $\mathcal{H}(\zeta, v) - c\mathcal{I}(\zeta, v)$ if and only if ζ is a critical point of $\mathcal{H}(\zeta, v_{c,\zeta}) - c\mathcal{I}(\zeta, v_{c,\zeta})$ and $v = v_{c,\zeta}$. We thus define

$$\mathcal{E}(\zeta) \overset{\text{def}}{=} \mathcal{I}(\zeta, \mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]\zeta) = \gamma\overline{\mathcal{E}}(\zeta) + \underline{\mathcal{E}}(\zeta), \tag{1.6}$$

where

$$\overline{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} \frac{\zeta^2}{1-\zeta} + \frac{1}{3}(1-\zeta)^3 \big(\partial_x \mathsf{F}_1\{\frac{\zeta}{1-\zeta}\}\big)^2 \mathrm{d}x,$$
$$\underline{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} \frac{\zeta^2}{\delta^{-1}+\zeta} + \frac{1}{3}(\delta^{-1}+\zeta)^3 \big(\partial_x \mathsf{F}_2\{\frac{\zeta}{\delta^{-1}+\zeta}\}\big)^2 \mathrm{d}x,$$

and look for critical points of $\mathcal{H}(\zeta, v_{c,\zeta}) - c\mathcal{I}(\zeta, v_{c,\zeta})$ by considering the minimization problem

$$\arg\min \Big\{ \mathcal{E}(\zeta),\ (\gamma+\delta)\big\|\zeta\big\|_{L^2}^2 = q \Big\}, \tag{1.7}$$

with c^{-2} acting as a Lagrange multiplier. Indeed, the corresponding Euler–Lagrange equation reads

$$2(\gamma+\delta)\zeta = c^2\ \mathrm{d}\mathcal{E}(\zeta) = 2c^2\mathcal{A}^{\mathsf{F}}_{\gamma,\delta}[\zeta]\zeta - c^2\frac{h_1^2 - \gamma h_2^2}{h_1^2 h_2^2}\zeta^2 + 2c^2\mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,\zeta], \tag{1.8}$$

which is obviously equivalent to (1.3), with $w = c\zeta$.

1.3. Statement of the Results

For the sake of readability, we postpone to Sect. 2 the definition and (standard) notations of the functional spaces used herein. The class of Fourier multipliers for which our main result is valid is the following.

Definition 1.1. (*Admissible class of Fourier multipliers*)

(i) $\mathsf{F}(k) = \mathsf{F}(|k|)$ and $0 < \mathsf{F} \leq 1$;
(ii) $\mathsf{F} \in \mathcal{C}^2(\mathbb{R})$, $\mathsf{F}(0) = 1$ and $\mathsf{F}'(0) = 0$;
(iii) There exists an integer $j \geq 2$ such that

$$\partial_k^j(k\mathsf{F}(k)) \in L^2(\mathbb{R});$$

(iv) There exists $\theta \in [0, 1)$ and $C^{\mathsf{F}}_{\pm} > 0$ such that

$$C^{\mathsf{F}}_{-}(1+|k|)^{-\theta} \leq \mathsf{F}(k) \leq C^{\mathsf{F}}_{+}(1+|k|)^{-\theta}.$$

We also introduce a second class of strongly admissible Fourier multipliers which is used in our second result.

Definition 1.2. (*Strongly admissible class of Fourier multipliers*) An admissible Fourier multipler F in the sense of Definition 1.1 is strongly admissible if $\mathsf{F} \in \mathcal{C}^\infty(\mathbb{R})$ and for each $j \in \mathbb{N}$ there exists a constant C_j such that

$$|\partial_k^j \mathsf{F}(k)| \leq C_j(1+|k|)^{-\theta-j}.$$

Notice the following.

Proposition 1.3. *The two aforementioned examples, namely* $\mathsf{F}_i^{\mathrm{id}}$ *and* $\mathsf{F}_i^{\mathrm{imp}}$ *are strongly admissible, and satisfy Definition 1.1(iv) with (respectively)* $\theta = 0, 1/2$.

Assumption 1.4. (Admissible parameters) *In the following, we fix* $\gamma \geq 0$, $\delta \in (0, \infty)$ *such that* $\delta^2 - \gamma \neq 0$. *We also fix a positive number* ν *such that* $\nu \geq 1 - \theta$ *and* $\nu > 1/2$ *(the second condition is automatically satisfied if* $\theta < 1/2$*). Finally, fix* R *an arbitrary positive constant.*

Remark 1.5. Our results hold for any values of the parameters $(\gamma,\delta)\in[0,\infty)\times(0,\infty)$ such that $\delta^2\neq\gamma$, although admissible values for q_0 depend on the choice of the parameters. However, not all parameters are physically relevant in the oceanographic context. When $\gamma>1$, the upper fluid is heavier than the lower fluid, and the system suffers from strong Rayleigh–Taylor instabilities [12]. In the bilayer setting, the use of the rigid-lid assumption is well-grounded only when the density contrast, $1-\gamma$, is small. In this situation, one may use the Boussinesq approximation, that is set $\gamma=1$; see [22] in the dispersionless setting. Notice however that system (1.2) exhibits unstable modes that are reminiscent of Kelvin–Helmholtz instabilities when the Fourier multipliers F_i satisfy Definition 1.1(iv) with $\theta\in[0,1)$; see [23]. It is therefore noteworthy that internal solitary waves in the ocean and in laboratory experiments are remarkably stable and fit very well with the Miyata–Choi–Camassa predictions [27]. The sign of the parameter $\delta^2-\gamma$ is known to determine whether long solitary waves are of elevation or depression type, as corroborated by Theorem 1.7. At the critical value $\delta^2=\gamma$, the first-order model would be the modified (cubic) KdV equation, predicting that no solitary wave exists [21].

We study the constrained minimization problem

$$\arg\min_{\zeta\in V_{q,R}}\mathcal{E}(\zeta), \tag{1.9}$$

with

$$V_{q,R}=\{\zeta\in H^\nu(\mathbb{R})\ :\ \|\zeta\|_{H^\nu(\mathbb{R})}<R,\ (\gamma+\delta)\|\zeta\|_{L^2(\mathbb{R})}^2=q\},$$

and $q\in(0,q_0)$, with q_0 sufficiently small. Notice in particular that as soon as q is sufficiently small $\|\zeta\|_{L^\infty}<\min(1,\delta^{-1})$ (by Lemma 2.1 thereafter and since $\nu>1/2$) and $\mathcal{E}(\zeta)$ is well-defined (by Lemmas 2.3 and 2.4 and since $\nu\geq 1-\theta$) for any $\zeta\in V_{q,R}$. Any solution will satisfy the Euler–Lagrange equation

$$\mathrm{d}\mathcal{E}(\zeta)+2\alpha(\gamma+\delta)\zeta=0, \tag{1.10}$$

where α is a Lagrange multiplier. Equation (1.10) is exactly (1.8) with $(-\alpha)^{-1}=c^2$, and therefore provides a traveling-wave solution to (1.2).

Our goal is to prove the following theorems.

Theorem 1.6. *Let γ,δ,ν,R satisfying Assumption 1.4 and F_i, $i=1,2$ be admissible in the sense of Definition 1.1. Let $D_{q,R}$ be the set of minimizers of $\mathcal{E}$ over $V_{q,R}$. Then there exists $q_0>0$ such that for all $q\in(0,q_0)$, the following statements hold:*

- *The set $D_{q,R}$ is nonempty and each element in $D_{q,R}$ solves the traveling wave equation (1.8), with $c^2=(-\alpha)^{-1}>1$. Thus for any $\zeta\in D_{q,R}$, $\big(\zeta(x\pm ct),w_\pm=\pm c\zeta(x\pm ct)\big)$ is a supercritical solitary wave solution to (1.2).*
- *For any minimizing sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ for $\mathcal{E}$ in $V_{q,R}$ such that $\sup_{n\in\mathbb{N}}\|\zeta_n\|_{H^\nu(\mathbb{R})}<R$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that a subsequence of $\{\zeta_n(\cdot+x_n)\}_{n\in\mathbb{N}}$ converges (strongly in $H^\nu(\mathbb{R})$ if $\nu=1-\theta>1/2$; weakly in $H^\nu(\mathbb{R})$ and strongly in $H^s(\mathbb{R})$ for $s\in[0,\nu)$ otherwise) to an element in $D_{q,R}$.*
- *There exist two constants $m,M>0$ such that*

$$\|\zeta\|_{H^\nu(\mathbb{R})}^2\leq Mq\quad\text{ and }\quad c^{-2}=-\alpha\leq 1-mq^{\frac{2}{3}},$$

  uniformly over $q\in(0,q_0)$ and $\zeta\in D_{q,R}$.

Theorem 1.7. *In addition to the hypotheses of Theorem 1.6, assume that F_i, $i=1,2$, are strongly admissible in the sense of Definition 1.2. Then there exists $q_0>0$ such that for any $q\in(0,q_0)$, each $\zeta\in D_{q,R}$ belongs to $H^s(\mathbb{R})$ for any $s\geq 0$ and*

$$\sup_{\zeta\in D_{q,R}}\inf_{x_0\in\mathbb{R}}\|q^{-\frac{2}{3}}\zeta(q^{-1/3}\cdot)-\xi_{\mathrm{KdV}}(\cdot-x_0)\|_{H^1(\mathbb{R})}=\mathcal{O}(q^{\frac{1}{6}})$$

where

$$\xi_{\mathrm{KdV}}(x) = \frac{\alpha_0(\gamma+\delta)}{\delta^2-\gamma}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{3\alpha_0(\gamma+\delta)}{\gamma+\delta^{-1}}}x\right)$$

is the unique (up to translation) solution of the KdV *equation* (5.2) *and*

$$\alpha_0 = \frac{3}{4}\left(\frac{(\delta^2-\gamma)^4}{(\gamma+\delta)^4(\gamma+\delta^{-1})}\right)^{\frac{1}{3}}.$$

In addition, the number α*, defined in Theorem 1.6, satisfies*

$$\alpha + 1 = q^{\frac{2}{3}}\alpha_0 + \mathcal{O}(q^{\frac{5}{6}}),$$

uniformly over $q \in (0, q_0)$ *and* $\zeta \in D_{q,R}$.

2. Technical Results

In the following, we denote $C(\lambda_1, \lambda_2, \dots)$ a positive constant depending non-decreasingly on the parameters $\lambda_1, \lambda_2, \dots$. We write $A \lesssim B$ when $A \le CB$ with C a nonnegative constant whose value is of no importance. We do not display the dependence with respect to the parameters $\gamma, \delta, C_{\pm}^{\mathsf{F}_i}$ and regularity indexes.

2.1. Functional Setting on the Real Line

Here and thereafter, we denote $L^2(\mathbb{R})$ the standard Lebesgue space of square-integrable functions, endowed with the norm $\|f\|_{L^2} = (\int_{-\infty}^{\infty}|f(x)|^2 \,\mathrm{d}x)^{1/2}$. The real inner product of $f_1, f_2 \in L^2(\mathbb{R})$ is denoted by $\langle f_1, f_2\rangle = \int_{\mathbb{R}} f_1(x)f_2(x)\mathrm{d}x$. We use the same notation for duality pairings which are clear from the context. The space $L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue-measurable functions f, endowed with the norm $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)|$. For any real constant $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with finite norm $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2} < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. For $n \in \mathbb{N}$, $\mathcal{C}^n(\mathbb{R})$ is the space of functions having continuous derivatives up to order n, and $\mathcal{C}^\infty(\mathbb{R}) = \bigcap_{n\in\mathbb{N}}\mathcal{C}^n(\mathbb{R})$. The Schwartz space is denoted $\mathcal{S}(\mathbb{R})$ and the tempered distributions $\mathcal{S}'(\mathbb{R})$. We use the following convention for the Fourier transform:

$$\mathcal{F}(f)(k) = \hat{f}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-\mathrm{i}xk}\,\mathrm{d}x.$$

We start with standard estimates in Sobolev spaces. The following interpolation estimates are standard and used without reference in our proofs.

Lemma 2.1. (Interpolation estimates) *Let* $f \in H^\mu(\mathbb{R})$*, with* $\mu > 1/2$.

(i) *One has* $f \in L^\infty(\mathbb{R})$ *and*

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{1-\frac{1}{2\mu}}\|f\|_{H^\mu}^{\frac{1}{2\mu}}.$$

(ii) *For any* $\delta \in (0,\mu)$*, one has* $f \in H^{\mu-\delta}(\mathbb{R})$ *and*

$$\|f\|_{H^{\mu-\delta}} \le \|f\|_{L^2}^{\frac{\delta}{\mu}}\|f\|_{H^\mu}^{1-\frac{\delta}{\mu}}.$$

The following lemma is given for instance in [5, Theorem C.12].

Lemma 2.2. (Composition estimate) *Let* G *be a smooth function vanishing at* 0*, and* $f \in H^\mu(\mathbb{R})$ *with* $\mu > 1/2$*. Then* $G\circ f \in H^\mu(\mathbb{R})$ *and we have*

$$\|G\circ f\|_{H^\mu} \le C(\|f\|_{L^\infty})\|f\|_{H^\mu}.$$

Lemma 2.3. (Product estimates)

(i) *For any* $f,g\in L^\infty(\mathbb{R})\cap H^s(\mathbb{R})$ *with* $s\geq 0$, *one has* $fg\in H^s(\mathbb{R})$ *and*
$$\big\|fg\big\|_{H^s}\lesssim \big\|f\big\|_{H^s}\big\|g\big\|_{L^\infty}+\big\|g\big\|_{H^s}\big\|f\big\|_{L^\infty}.$$

(ii) *For any* $f\in H^s(\mathbb{R}), g\in H^t(\mathbb{R})$ *with* $s+t\geq 0$, *and let* r *such that* $\min(s,t)\geq r$ *and* $r<s+t-1/2$. *Then one has* $fg\in H^r(\mathbb{R})$ *and*
$$\big\|fg\big\|_{H^r}\lesssim \big\|f\big\|_{H^s}\big\|g\big\|_{H^t}.$$

(iii) *For any* $\zeta\in L^\infty(\mathbb{R})$ *such that* $\big\|\zeta\big\|_{L^\infty}\leq 1-h_0$ *with* $h_0>0$ *and any* $f\in L^\infty(\mathbb{R})$, *one has*
$$\big\|\frac{f}{1+\zeta}\big\|_{L^\infty}\leq C(h_0^{-1})\big\|f\big\|_{L^\infty}.$$

(iv) *For any* $\zeta\in H^\mu(\mathbb{R})$ *with* $\mu>1/2$ *such that* $\big\|\zeta\big\|_{L^\infty}\leq 1-h_0$ *with* $h_0>0$ *and any* $f\in H^s(\mathbb{R})$ *with* $s\in[-\mu,\mu]$, *one has*
$$\big\|\frac{f}{1+\zeta}\big\|_{H^s}\leq C(h_0^{-1},\big\|\zeta\big\|_{H^\mu})\big\|f\big\|_{H^s}.$$

Proof. The first two items are standard (see for instance [5, Prop. C.11 and Th. C.10]). The third item is obvious. The last item is proved using *(ii)* and Lemma 2.2. □

The following lemma justifies the assumptions of admissible Fourier multipliers in Definition 1.1.

Lemma 2.4. (Properties of admissible Fourier multipliers) *Any admissible Fourier multipler (in the sense of Definition 1.1),* F_i, *satisfies the following.*

(i) *The linear operator* $\partial_x\mathsf{F}_i(D)$ *is bounded from* $H^s(\mathbb{R})$ *to* $H^{s-1+\theta}(\mathbb{R})$, *for any* $s\in\mathbb{R}$, *and*
$$\big\|\partial_x\mathsf{F}_i\big\|_{H^s\to H^{s-1+\theta}}\lesssim C_+^{\mathsf{F}_i}.$$
Moreover, for any $\zeta\in H^{s+1-\theta}$, *one has*
$$\big\|\zeta\big\|_{H^s}^2+(C_+^{\mathsf{F}_i})^{-2}\big\|\partial_x\mathsf{F}_i\{\zeta\}\big\|_{H^s}^2\lesssim\big\|\zeta\big\|_{H^{s+1-\theta}}^2\lesssim\big\|\zeta\big\|_{H^s}^2+(C_-^{\mathsf{F}_i})^{-2}\big\|\partial_x\mathsf{F}_i\{\zeta\}\big\|_{H^s}^2.$$

(ii) *Let* $\varphi\in\mathcal{C}^\infty(\mathbb{R})$ *with compact support and* $[\partial_x\mathsf{F}_i,\varphi]\zeta=\partial_x\mathsf{F}_i\{\varphi\zeta\}-\varphi\partial_x\mathsf{F}_i\{\zeta\}$. *Then*
$$\big\|[\partial_x\mathsf{F}_i,\varphi]\zeta\big\|_{L^2}\lesssim\big\|\widehat{\varphi'}\big\|_{L^1}\big\|\zeta\big\|_{H^{1-\theta}}.$$

(iii) *There exists* $j\geq 2$ *and* C_j *such that for any* $\zeta\in L^2(\mathbb{R})$ *with compact support*
$$|\partial_x\mathsf{F}_i\{\zeta\}|(x)\leq\frac{C_j}{\operatorname{dist}(x,\operatorname{supp}(\zeta))^j}\big\|\zeta\big\|_{L^2},\quad \textit{for a.a. } x\in\mathbb{R}\setminus\operatorname{supp}(\zeta).$$

Proof. The first result is obvious from Definition 1.1(i) and the definition of Sobolev spaces. For the second, we shall first prove that the function $\mathsf{G}_i:k\mapsto k\mathsf{F}_i(k)$ satisfies
$$|\mathsf{G}_i'(k)|\lesssim\langle k\rangle^{1-\theta}.\tag{2.1}$$
To this aim, let us first consider $\mathsf{G}\in\mathcal{S}(\mathbb{R})$ and χ a smooth cut-off function, such that $\chi(k)=1$ for $|k|\leq 1/2$ and $\chi(k)=0$ for $|k|\geq 1$. We decompose
$$|\mathsf{G}'|(k)\leq|\chi(D)\mathsf{G}'|(k)+|(1-\chi(D))\mathsf{G}'|(k).$$
For the first contribution, one has
$$|\chi(D)\mathsf{G}'|(k)=\frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}}\hat\chi(\xi)\mathsf{G}'(k+\xi)\mathrm{d}\xi\right|\lesssim\sup_{\xi\in\mathbb{R}}\frac{|\mathsf{G}(k+\xi)|}{\langle k+\xi\rangle^{1-\theta}}\langle k\rangle^{1-\theta}\big\|\langle\cdot\rangle^{1-\theta}\hat\chi'\big\|_{L^1},$$
and the second contribution satisfies for any $j\geq 2$,
$$|(1-\chi(D))\mathsf{G}'|(k)\lesssim\big\|(1-\chi(\xi))|\xi|\hat{\mathsf{G}}(\xi)\big\|_{L^1}\lesssim\big\|\langle\xi\rangle^{-(j-1)}|\xi|^j\hat{\mathsf{G}}(\xi)\big\|_{L^1}\lesssim\big\|\mathsf{G}^{(j)}\big\|_{L^2},$$

by the Cauchy–Schwarz inequality and Parseval's theorem. Thus we find, for any $j \geq 2$,

$$|\mathsf{G}'|(k) \lesssim \big\|\langle\cdot\rangle^{\theta-1}\mathsf{G}\big\|_{L^\infty}\langle k\rangle^{1-\theta} + \big\|\mathsf{G}^{(j)}\big\|_{L^2}.$$

The same estimate applies to $\mathsf{G}(k) = k\mathsf{F}_i(k)$ by smooth approximation, and (2.1) follows from Definition 1.1. Using (2.1) and the mean value theorem together with Young's inequality, we find

$$\big\|[\partial_x\mathsf{F}_i,\varphi]\zeta\big\|_{L^2} \lesssim \big\|\widehat{\varphi'}\big\|_{L^1}\big\|(1+|\cdot|)^{1-\theta}\hat{\zeta}\big\|_{L^2} \lesssim \big\|\widehat{\varphi'}\big\|_{L^1}\big\|\zeta\big\|_{H^{1-\theta}}.$$

For the third result, let us assume at first that the kernel $K_i \stackrel{\text{def}}{=} \mathcal{F}^{-1}(ik\mathsf{F}_i(k)) \in L^2(\mathbb{R})$. Then one has

$$|\partial_x\mathsf{F}_i\{\zeta\}|(x) = \frac{1}{\sqrt{2\pi}}\left|\int_{\operatorname{supp}(\zeta)}\frac{(x-y)^jK_i(x-y)\zeta(y)}{(x-y)^j}\,\mathrm{d}y\right| \leq \frac{\big(|K_{i,j}|*|\zeta|\big)(x)}{\sqrt{2\pi}\operatorname{dist}(x,\operatorname{supp}(\zeta))^j} \lesssim \frac{\big\|\zeta\big\|_{L^2}}{\operatorname{dist}(x,\operatorname{supp}(\zeta))^j},$$

where we denote $K_{i,j}(x) = x^jK_i(x)$, remark that $K_{i,j} \in L^2(\mathbb{R})$ by Definition 1.1(iii) and Plancherel's theorem, and apply the Cauchy–Schwarz inequality to the convolution. If $K_i \notin L^2(\mathbb{R})$, we obtain the result by regularizing K_i (*i.e.* smoothly truncating F_i) and passing to the limit. □

Lemma 2.5. *Let $\gamma \geq 0$, $\delta > 0$, $\mu > 1/2$ and F_i be admissible Fourier multipliers. Assume that $\zeta \in H^\mu(\mathbb{R})$ is such that $1 - \big\|\zeta\big\|_{L^\infty} \geq h_0$, $\delta^{-1} - \big\|\zeta\big\|_{L^\infty} \geq h_0$, with $h_0 > 0$. Then there exist a constant $C_0 = C(h_0^{-1}, \big\|\zeta\big\|_{H^\mu})$ such that*

$$C_0^{-1}\big\|\zeta\big\|_{H^{1-\theta}}^2 \leq \mathcal{E}(\zeta) \leq C_0\big\|\zeta\big\|_{H^{1-\theta}}^2.$$

Proof. We first deal with the contribution of $\overline{\mathcal{E}}(\zeta)$ defined in (1.6). By Lemma 2.4(i) we get that

$$\overline{\mathcal{E}}(\zeta) \leq C(\big\|\zeta\big\|_{L^\infty})\big\|\frac{\zeta}{1-\zeta}\big\|_{H^{1-\theta}}^2 \quad\text{and}\quad \big\|\frac{\zeta}{1-\zeta}\big\|_{H^{1-\theta}}^2 \leq C(h_0^{-1})\overline{\mathcal{E}}(\zeta).$$

By Lemma 2.3(iv), one has

$$\big\|\frac{\zeta}{1-\zeta}\big\|_{H^{1-\theta}} \leq C(h_0^{-1}, \big\|\zeta\big\|_{H^\mu})\big\|\zeta\big\|_{H^{1-\theta}},$$

and the triangle inequality together with Lemma 2.3(ii) yields

$$\big\|\zeta\big\|_{H^{1-\theta}} \lesssim \big\|\frac{\zeta}{1-\zeta}\big\|_{H^{1-\theta}} + \big\|\frac{\zeta^2}{1-\zeta}\big\|_{H^{1-\theta}} \lesssim \big\|\frac{\zeta}{1-\zeta}\big\|_{H^{1-\theta}} + \big\|\zeta\big\|_{H^\mu}\big\|\frac{\zeta}{1-\zeta}\big\|_{H^{1-\theta}}.$$

Collecting the above information, we find that

$$C_0^{-1}\big\|\zeta\big\|_{H^{1-\theta}}^2 \leq \overline{\mathcal{E}}(\zeta) \leq C_0\big\|\zeta\big\|_{H^{1-\theta}}^2,$$

with $C_0 = C(h_0^{-1}, \big\|\zeta\big\|_{H^\mu})$. Similar estimates hold for $\underline{\mathcal{E}}(\zeta)$, and thus for $\mathcal{E}(\zeta) = \gamma\overline{\mathcal{E}}(\zeta) + \underline{\mathcal{E}}(\zeta)$. □

Lemma 2.6. *Let $\gamma \geq 0$, $\delta > 0$, $\mu > 1/2$ and F_i be admissible Fourier multipliers. Assume that, for $j \in \{1,2\}$, $\zeta_j \in H^\mu(\mathbb{R})$ is such that $1 - \big\|\zeta_j\big\|_{L^\infty} \geq h_0$ and $\delta^{-1} - \big\|\zeta_j\big\|_{L^\infty} \geq h_0$, with $h_0 > 0$. Then one has*

$$\mathcal{E}(\zeta_1) - \mathcal{E}(\zeta_2) \leq C(h_0^{-1}, \big\|\zeta_1\big\|_{H^\mu}, \big\|\zeta_2\big\|_{H^\mu})\big\|\zeta_1 - \zeta_2\big\|_{H^\mu}.$$

Proof. As previously, we detail the result for $\overline{\mathcal{E}}(\zeta)$, as the similar estimate for $\underline{\mathcal{E}}(\zeta)$ is obtained in the same way. One has

$$\begin{aligned}\overline{\mathcal{E}}(\zeta_1) - \overline{\mathcal{E}}(\zeta_2) = \int_{\mathbb{R}} \frac{\zeta_1^2}{1-\zeta_1} - \frac{\zeta_2^2}{1-\zeta_2} + \frac{1}{3}\big[(1-\zeta_1)^3 - (1-\zeta_2)^3\big]\big(\partial_x\mathsf{F}_1\{\frac{\zeta_1}{1-\zeta_1}\}\big)^2 \\ +\frac{1}{3}(1-\zeta_2)^3\Big[\big(\partial_x\mathsf{F}_1\{\frac{\zeta_1}{1-\zeta_1}\}\big)^2 - \big(\partial_x\mathsf{F}_1\{\frac{\zeta_2}{1-\zeta_2}\}\big)^2\Big]\mathrm{d}x,\end{aligned}$$

By Lemma 2.3(iii), and the Cauchy–Schwarz inequality, we immediately have

$$\int_{\mathbb{R}} \left| \frac{\zeta_1^2}{1-\zeta_1} - \frac{\zeta_2^2}{1-\zeta_2} \right| \mathrm{d}x \le C(h_0^{-1}, \|\zeta_1\|_{L^\infty}, \|\zeta_2\|_{L^\infty})(\|\zeta_1\|_{L^2} + \|\zeta_2\|_{L^2})\|\zeta_1-\zeta_2\|_{L^2}.$$

Similarly, we find by Lemmas 2.4(i), 2.3(iv),

$$\int_{\mathbb{R}} \left| \big[(1-\zeta_1)^3 - (1-\zeta_2)^3\big]\big(\partial_x \mathsf{F}_1\{\frac{\zeta_1}{1-\zeta_1}\}\big)^2 \right| \mathrm{d}x \le C(h_0^{-1}, \|\zeta_1\|_{H^\mu}, \|\zeta_2\|_{L^\infty})\|\zeta_1-\zeta_2\|_{L^\infty}.$$

Finally we conclude by Lemma 2.3(iv) that

$$\begin{aligned}
&\int_{\mathbb{R}} \left| (1-\zeta_2)^3 \Big[\big(\partial_x \mathsf{F}_1\{\frac{\zeta_1}{1-\zeta_1}\}\big)^2 - \big(\partial_x \mathsf{F}_1\{\frac{\zeta_2}{1-\zeta_2}\}\big)^2\Big] \right| \mathrm{d}x \\
&\quad \le C(\|\zeta_2\|_{L^\infty})\Big\|\frac{\zeta_1}{1-\zeta_1} - \frac{\zeta_2}{1-\zeta_2}\Big\|_{H^{1-\theta}}\Big\|\frac{\zeta_1}{1-\zeta_1} + \frac{\zeta_2}{1-\zeta_2}\Big\|_{H^{1-\theta}} \\
&\quad \le C(h_0^{-1}, \|\zeta_1\|_{H^\mu}, \|\zeta_2\|_{H^\mu})\|\zeta_1-\zeta_2\|_{H^{1-\theta}},
\end{aligned}$$

The result is proved. □

Lemma 2.7. *Let $\gamma \ge 0$, $\delta > 0$, and F_i be admissible Fourier multipliers. Let $l \in \{1,2,3\}$ and $\zeta \in H^l(\mathbb{R})$ such that $1 - \|\zeta\|_{L^\infty} \ge h_0$ and $\delta^{-1} - \|\zeta\|_{L^\infty} \ge h_0$, with $h_0 > 0$. Then one can decompose*

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} (\gamma+\delta)\zeta^2 + (\gamma-\delta^2)\zeta^3 + \frac{\gamma+\delta^{-1}}{3}(\partial_x\zeta)^2 \ \mathrm{d}x + \mathcal{E}_{\mathrm{rem}}(\zeta),$$

and

$$\langle \mathrm{d}\mathcal{E}(\zeta), \zeta\rangle = \int_{\mathbb{R}} 2(\gamma+\delta)\zeta^2 + 3(\gamma-\delta^2)\zeta^3 + 2\frac{\gamma+\delta^{-1}}{3}(\partial_x\zeta)^2 \ \mathrm{d}x + \langle \mathrm{d}\mathcal{E}_{\mathrm{rem}}(\zeta), \zeta\rangle,$$

where

$$|\mathcal{E}_{\mathrm{rem}}| + |\langle \mathrm{d}\mathcal{E}_{\mathrm{rem}}(\zeta), \zeta\rangle| \le C(h_0^{-1}, \|\zeta\|_{H^1})(\|\zeta\|_{L^\infty}^2\|\zeta\|_{L^2}^2 + \|\zeta\|_{L^\infty}\|\partial_x\zeta\|_{L^2}^2 + \|\partial_x^l\zeta\|_{L^2}\|\partial_x\zeta\|_{L^2}).$$

Proof. We consider $\overline{\mathcal{E}}(\zeta)$; the corresponding expansion for $\underline{\mathcal{E}}(\zeta)$ is obtained similarly. We write

$$\overline{\mathcal{E}}(\zeta) = \int_{\mathbb{R}} \zeta^2 + \zeta^3 + \frac{1}{3}(\partial_x\zeta)^2 \ \mathrm{d}x + \overline{\mathcal{E}}_{\mathrm{rem}}(\zeta),$$

where

$$\begin{aligned}
\overline{\mathcal{E}}_{\mathrm{rem}}(\zeta) = &\int_{\mathbb{R}} \frac{\zeta^4}{1-\zeta}\,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}} (1-\zeta)^3\big(\partial_x\{\frac{\zeta}{1-\zeta}\}\big)^2 - (\partial_x\zeta)^2\,\mathrm{d}x \\
&+ \int_{\mathbb{R}} (1-\zeta)^3 \Big[\big(\partial_x \mathsf{F}_1\{\frac{\zeta}{1-\zeta}\}\big)^2 - \big(\partial_x\{\frac{\zeta}{1-\zeta}\}\big)^2\Big]\,\mathrm{d}x.
\end{aligned}$$

The first integral is bounded by $h_0^{-1}\|\zeta\|_{L^\infty}^2\|\zeta\|_{L^2}^2$ and the second one by $h_0^{-1}\|\zeta\|_{L^\infty}\|\partial_x\zeta\|_{L^2}^2$. Moreover

$$\begin{aligned}
&\left|\int_{\mathbb{R}} (1-\zeta)^3 \left[\left(\partial_x \mathsf{F}_1\left\{\frac{\zeta}{1-\zeta}\right\}\right)^2 - \left(\partial_x\left\{\frac{\zeta}{1-\zeta}\right\}\right)^2\right]\mathrm{d}x\right| \\
&\quad \le \int_{\mathbb{R}} (1+|\zeta|)^3 \left|(\partial_x\mathsf{F}_1 - \partial_x)\left(\frac{\zeta}{1-\zeta}\right)\right| \left|(\partial_x\mathsf{F}_1 + \partial_x)\left(\frac{\zeta}{1-\zeta}\right)\right| \mathrm{d}x.
\end{aligned}$$

Applying the Cauchy–Schwarz inequality, Plancherel's theorem and the estimates

$$|\mathsf{F}_1(k) - 1| \lesssim |k|^{l-1}, \qquad |\mathsf{F}_1(k)+1| \lesssim 1,$$

(by Definition 1.1, (i and ii)), we deduce

$$\begin{aligned}\Big|\int_{\mathbb{R}}(1-\zeta)^3\Big[\Big(\partial_x\mathsf{F}_1\Big\{\frac{\zeta}{1-\zeta}\Big\}\Big)^2-\Big(\partial_x\Big\{\frac{\zeta}{1-\zeta}\Big\}\Big)^2\Big]\,\mathrm{d}x\Big|\\ \leq(1+\|\zeta\|_{L^\infty})^3\|\partial_x^l\Big(\frac{\zeta}{1-\zeta}\Big)\|_{L^2}\|\partial_x\Big(\frac{\zeta}{1-\zeta}\Big)\|_{L^2}\\ \leq C(\|\zeta\|_{H^\mu})\|\partial_x^l\zeta\|_{L^2}\|\partial_x\zeta\|_{L^2},\end{aligned}$$

where the last inequality follows from Leibniz's rule and standard bilinear estimates [5, Prop. C.12]. Combining the above estimates together with similar calculations for $\underline{\mathcal{E}}$ yields the desired estimate for $|\mathcal{E}_{\mathrm{rem}}|$. The estimate for $|\langle\mathrm{d}\mathcal{E}_{\mathrm{rem}}(\zeta),\zeta\rangle|$ follows in the same way when decomposing

$$\begin{aligned}\langle\mathrm{d}\mathcal{E}(\zeta),\zeta\rangle=\int_{\mathbb{R}}2\frac{h_1+\gamma h_2}{h_1h_2}\zeta^2-\frac{h_1^2-\gamma h_2^2}{h_1^2h_2^2}\zeta^3+\frac{2}{3}\delta^{-1}h_2^3\big(\partial_x\mathsf{F}_2\{h_2^{-1}\zeta\}\big)\big(\partial_x\mathsf{F}_2\{h_2^{-2}\zeta\}\big)\\ +\frac{2\gamma}{3}h_1^3\big(\partial_x\mathsf{F}_1\{h_1^{-1}\zeta\}\big)\big(\partial_x\mathsf{F}_1\{h_1^{-2}\zeta\}\big)+\zeta\big(h_2\partial_x\mathsf{F}_2\{h_2^{-1}\zeta\}\big)^2-\gamma\zeta\big(h_1\partial_x\mathsf{F}_1\{h_1^{-1}\zeta\}\big)^2\,\mathrm{d}x,\end{aligned}\tag{2.2}$$

and we do not detail for the sake of conciseness. □

Using very similar arguments we obtain the following alternative decomposition.

Lemma 2.8. *Let $\gamma\geq 0$, $\delta>0$, $\mu>1/2$ and F_i be admissible Fourier multipliers such that $\mu\geq 1-\theta$. Let $\zeta\in H^\mu(\mathbb{R})$ such that $1-\|\zeta\|_{L^\infty}\geq h_0$ and $\delta^{-1}-\|\zeta\|_{L^\infty}\geq h_0$, with $h_0>0$. Then one can decompose*

$$\mathcal{E}(\zeta)=\mathcal{E}_2(\zeta)+\mathcal{E}_3(\zeta)+\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta)$$

and

$$\langle\mathrm{d}\mathcal{E}(\zeta),\zeta\rangle=2\mathcal{E}_2(\zeta)+3\mathcal{E}_3(\zeta)+\mathcal{E}_{\mathrm{rem}}^{(2)}(\zeta),$$

where

$$\begin{aligned}\mathcal{E}_2(\zeta)&=\int_{\mathbb{R}}(\gamma+\delta)\zeta^2+\gamma\frac{1}{3}(\partial_x\mathsf{F}_1\{\zeta\})^2+\delta^{-1}\frac{1}{3}(\partial_x\mathsf{F}_2\{\zeta\})^2\,\mathrm{d}x,\\ \mathcal{E}_3(\zeta)&=\int_{\mathbb{R}}(\gamma-\delta^2)\zeta^3-\gamma\zeta(\partial_x\mathsf{F}_1\{\zeta\})^2+\zeta(\partial_x\mathsf{F}_2\{\zeta\})^2\\ &\qquad+\gamma\frac{2}{3}(\partial_x\mathsf{F}_1\{\zeta\})(\partial_x\mathsf{F}_1\{\zeta^2\})-\frac{2}{3}(\partial_x\mathsf{F}_2\{\zeta\})(\partial_x\mathsf{F}_2\{\zeta^2\})\,\mathrm{d}x.\end{aligned}$$

Moreover, one has $\mathcal{E}_2(\zeta)\geq(\gamma+\delta)\|\zeta\|_{L^2}^2$ and

$$\begin{aligned}|\mathcal{E}_3(\zeta)|\leq C(h_0^{-1},\|\zeta\|_{H^\mu})\|\zeta\|_{L^\infty}\|\zeta\|_{H^{1-\theta}}^2,\\ \forall j\in\{1,2\},\quad|\mathcal{E}_{\mathrm{rem}}^{(j)}(\zeta)|\leq C(h_0^{-1},\|\zeta\|_{H^\mu})\|\zeta\|_{L^\infty}^2\|\zeta\|_{H^{1-\theta}}^2.\end{aligned}$$

2.2. Periodic Functional Setting

Given $P>0$, we denote L^2_P the space of P-periodic, locally square-integrable functions, endowed with the norm

$$\|u\|_{L^2_P}=\|u\|_{L^2(-P/2,P/2)}\stackrel{\mathrm{def}}{=}\Big(\int_{-P/2}^{P/2}|u(x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}.$$

The Fourier coefficients of $u\in L^2_P$ are defined by

$$\hat{u}_k\stackrel{\mathrm{def}}{=}\frac{1}{\sqrt{P}}\int_{-P/2}^{P/2}u(x)e^{-\frac{2\mathrm{i}\pi kx}{P}}\,\mathrm{d}x,\qquad u(x)=\frac{1}{\sqrt{P}}\sum_{k\in\mathbb{Z}}\hat{u}_ke^{\frac{2\mathrm{i}\pi kx}{P}}.$$

We define, for $s \geq 0$,

$$H^s_P \stackrel{\text{def}}{=} \left\{ u \in L^2_P, \quad \big\|u\big\|^2_{H^s_P} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \left(1 + \frac{4\pi^2 k^2}{P^2}\right)^s |\hat{u}_k|^2 < \infty \right\}.$$

The Fourier multiplier operator $\Lambda\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is defined as usual by $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. It maps periodic distributions to periodic distributions and we have

$$\widehat{\Lambda u}_k = \left(1 + \frac{4\pi^2 k^2}{P^2}\right)^{\frac{1}{2}} \hat{u}_k.$$

Thus

$$\big\|u\big\|^2_{H^s_P} = \int_{-P/2}^{P/2} u \Lambda^{2s} u \, \mathrm{d}x$$

and Λ^m is an isomorphism from H^s_P to H^{s-m}_P for any $s, m \in \mathbb{R}$. Similarly, the operators $\partial_x \mathsf{F}_i$ extend to operators from $\mathcal{S}'(\mathbb{R})$ to itself, and maps smoothly H^s_P into $H^{s-1+\theta}_P$, acting on the Fourier coefficients by pointwise multiplication:

$$\widehat{\partial_x \mathsf{F}_i u}_k = \frac{2\pi \mathrm{i} k}{P} \mathsf{F}_i(2\pi k/P) \hat{u}_k.$$

For any $s > 1/2$, the continuous embedding

$$\big\|u\big\|_{L^\infty} \leq \frac{1}{\sqrt{P}} \sum_{k\in\mathbb{Z}} |\hat{u}_k| \leq \big\|u\big\|_{H^s_P} \times \frac{1}{\sqrt{P}} \left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+\frac{4\pi^2 k^2}{P^2})^s} \right)^{\frac{1}{2}} \lesssim \big\|u\big\|_{H^s_P},$$

holds uniformly with respect to $P \geq 1$. More generally, one checks by a partition of unity argument, or repeating the proofs in the periodic setting, that Lemmas 2.1, 2.2, 2.3 and as a consequence Lemmas 2.5, 2.6, 2.7 and 2.8 have immediate analogues in the periodic setting, with uniform estimates with respect to $P \geq 1$, when defining

$$\mathcal{E}_P(\zeta) = \gamma \overline{\mathcal{E}}_P(\zeta) + \underline{\mathcal{E}}_P(\zeta)$$

where

$$\overline{\mathcal{E}}_P(\zeta) = \int_{-P/2}^{P/2} \frac{\zeta^2}{1-\zeta} + \frac{1}{3}(1-\zeta)^3 \big(\partial_x \mathsf{F}_1\{\frac{\zeta}{1-\zeta}\}\big)^2 \mathrm{d}x,$$
$$\underline{\mathcal{E}}_P(\zeta) = \int_{-P/2}^{P/2} \frac{\zeta^2}{\delta^{-1}+\zeta} + \frac{1}{3}(\delta^{-1}+\zeta)^3 \big(\partial_x \mathsf{F}_2\{\frac{\zeta}{\delta^{-1}+\zeta}\}\big)^2 \mathrm{d}x.$$

3. The Periodic Problem

Our first task is to construct periodic traveling-wave solutions with large periods by considering the periodic minimization problem corresponding to (1.9). We will use this in the next section to construct a special minimizing sequence for (1.9), which is useful when $\nu > 1-\theta$. When $\theta < 1/2$ and $\nu = 1-\theta$, any minimizing sequence has the special property and therefore it is strictly speaking unnecessary to first consider the periodic minimization problem. Nevertheless, we consider here all possible parameters in order to highlight some interesting differences between the cases $\nu = 1-\theta$ and $\nu > 1-\theta$.

We ensure that the hypotheses of Sect. 2, namely

$$\zeta \in H^\nu_P \quad \text{and} \quad \big\|\zeta\big\|_{L^\infty} < \min(1, \delta^{-1})$$

will be satisfied through a penalization argument. To this aim, we fix $R > 0$ and restrict ourselves to values $q \in (0, q_0)$ sufficiently small so that $\big\|\zeta\big\|_{H^\nu_P(\mathbb{R})} \leq 2R$ and $(\gamma+\delta)\big\|\zeta\big\|^2_{L^2_P} = q$ ensures (the reference to Lemma 2.1(i)) that $\big\|\zeta\big\|_{L^\infty} < \min(1, \delta^{-1}) - h_0$ with some $h_0 > 0$, uniformly with respect to $P \geq P_0$

sufficiently large (and likewise in the real line setting). We then define $\varrho : [0,(2R)^2) \to [0,\infty)$ a smooth, non-decreasing penalization function, satisfying

(i) $\varrho(t) = 0$ for $0 \le t \le R^2$;
(ii) $\varrho(t) \to \infty$ as $t \nearrow (2R)^2$;
(iii) For any $a_1 \in (0,1)$, there exists $M_1, M_2 > 0$ and $a_2 > 1$ such that

$$\varrho'(t) \le M_1 \varrho(t)^{a_1} + M_2 \varrho(t)^{a_2}; \tag{3.1}$$

for instance $\varrho\colon (R^2,(2R)^2) \ni t \mapsto ((2R)^2 - t)^{-1} \exp(\frac{1}{R^2-t})$.

Now consider the functional

$$\mathcal{E}_{P,\varrho}(\zeta) \stackrel{\text{def}}{=} \varrho(\|\zeta\|_{H^\nu_P}^2) + \mathcal{E}_P(\zeta)$$

and the constraint set

$$V_{P,q,2R} \stackrel{\text{def}}{=} \Big\{\zeta \in H^\nu_P, \quad (\gamma+\delta)\|\zeta\|_{L^2_P}^2 = q \text{ and } \|\zeta\|_{H^\nu_P} < 2R\Big\}.$$

Standard weak continuity arguments (see *e.g.* [43, §I.1,I.2]) yield the existence of a minimizer for $\mathcal{E}_{P,\varrho}$ when q is sufficiently small. More precisely, we have the following lemma.

Lemma 3.1. *There exists $q_0 > 0$ such that for any $q \in (0,q_0)$, the functional $\mathcal{E}_{P,\varrho} : V_{P,q,2R} \to \mathbb{R}$ is weakly lower semi-continuous, bounded from below and $\mathcal{E}_{P,\varrho}(\zeta) \to \infty$ as $\|\zeta\|_{H^\nu_P} \nearrow 2R$. In particular, it has a minimizer $\zeta_P \in V_{P,q,2R}$, which satisfies the Euler–Lagrange equation*

$$2\varrho'(\|\zeta_P\|_{H^\nu_P}^2)\Lambda^{2\nu}\zeta_P + \mathrm{d}\mathcal{E}_P(\zeta_P) + 2\alpha_P(\gamma+\delta)\zeta_P = 0 \tag{3.2}$$

for some Lagrange multiplier $\alpha_P(\zeta_P) \in \mathbb{R}$.

Now we wish to prove that $\zeta_P \in V_{P,q,R}$, and in particular satisfies the Euler–Lagrange equation

$$\mathrm{d}\mathcal{E}_P(\zeta_P) + 2\alpha_P(\gamma+\delta)\zeta_P = 0.$$

From this point on, we heavily make use of the property (see Assumption 1.4)

$$\gamma - \delta^2 \neq 0.$$

without explicit references in the statements.

Lemma 3.2. *There exists $m > 0$ and $q_0 > 0$ such that for any $q \in (0,q_0)$,*

$$I_q \stackrel{\text{def}}{=} \inf\{\mathcal{E}(\zeta),\ \zeta \in V_{q,R}\} < q(1 - mq^{\frac{2}{3}})$$

and there exists $P_q > 0$ such that

$$I_{P,\varrho,q} \stackrel{\text{def}}{=} \inf\{\mathcal{E}_{P,\varrho}(\zeta),\ \zeta \in V_{P,q,2R}\} < q(1 - mq^{\frac{2}{3}})$$

for any $P \ge P_q$.

Proof. Let us first consider the case of the real line. Consider $\psi \in \mathcal{C}^\infty(\mathbb{R})$ with compact support, such that $(\gamma+\delta)\|\psi\|_{L^2}^2 = 1$; and denote $\psi_\lambda : x \mapsto \lambda^{\frac12}\psi(\lambda x)$. One has

$$\int_{\mathbb{R}} \psi_\lambda^3 \,\mathrm{d}x = \lambda^{\frac12}\int_{\mathbb{R}} \psi^3 \,\mathrm{d}x \quad \text{and} \quad \|\partial_x\psi_\lambda\|_{L^2} = \lambda\|\partial_x\psi\|_{L^2}.$$

It follows that, for the case when $\gamma - \delta^2 < 0$, one can choose $\psi \ge 0$ and λ small enough so that

$$\int_{\mathbb{R}} (\gamma-\delta^2)\psi_\lambda^3 + \frac{\gamma+\delta^{-1}}{3}(\partial_x\psi_\lambda)^2 \,\mathrm{d}x \stackrel{\text{def}}{=} -2m < 0.$$

If $\gamma - \delta^2 > 0$, we instead let $\psi \le 0$ and again choose λ small enough so that the above holds.

Now, consider $\phi_q : x \mapsto q^{\frac23}\psi_\lambda(q^{\frac13}x)$. One has

$$\|\phi_q\|_{L^\infty} \le q^{\frac23}\|\psi_\lambda\|_{L^\infty}, \quad \int_{\mathbb{R}} \phi_q^3 \,\mathrm{d}x = q^{\frac53}\int_{\mathbb{R}} \psi_\lambda^3 \,\mathrm{d}x \quad \text{and} \quad \|\partial_x^n\phi_q\|_{L^2} = q^{\frac12+\frac{n}{3}}\|\partial_x^n\psi_\lambda\|_{L^2} \quad (n \in \mathbb{N}).$$

In particular, for q sufficiently small, $\|\phi_q\|_{H^\nu} < R$; and by Lemma 2.7 with $l = 3$,

$$\mathcal{E}(\phi_q) = (\gamma+\delta)\int_{\mathbb{R}} \phi_q^2 \,\mathrm{d}x + \int_{\mathbb{R}} (\gamma-\delta^2)\phi_q^3 + \frac{\gamma+\delta^{-1}}{3}(\partial_x\phi_q)^3 \,\mathrm{d}x + \mathcal{O}(q^{\frac{7}{3}})$$
$$= q - 2mq^{\frac{5}{3}} + \mathcal{O}(q^{\frac{7}{3}}).$$

The result follows in the real-line setting.

The result in the periodic setting is deduced in a similar way when restricting to $P \geq P_q$ sufficiently large so that $\mathrm{supp}(\phi_q) \in (-P/2, P/2)$, and considering $\phi_{P,q} = \sum_{j\in\mathbb{Z}} \phi_q(x - jP)$. □

Lemma 3.3. *There exists $q_0 > 0$ such that for any $q \in (0, q_0)$, one has*

$$\forall P \geq P_q, \qquad |\alpha_P + 1| < \frac{1}{2},$$

where α_P is defined in Lemma 3.1 and P_q in Lemma 3.2.

Proof. Testing the Euler–Lagrange equation (3.2) against ζ_P yields

$$2\varrho'(\|\zeta_P\|_{H_P^\nu}^2)\|\zeta_P\|_{H_P^\nu}^2 + 2\alpha_P(\gamma+\delta)\|\zeta_P\|_{L_P^2}^2 + \langle \mathrm{d}\mathcal{E}_P(\zeta_P), \zeta_P\rangle = 0.$$

Using the decompositions in Lemma 2.8 (in the periodic setting) yields

$$-\alpha_P\, q = \mathcal{E}_P(\zeta_P) + \frac{1}{2}\mathcal{E}_{3,P}(\zeta_P) + \frac{1}{2}\mathcal{E}^{(2)}_{\mathrm{rem},P}(\zeta_P) - \mathcal{E}^{(1)}_{\mathrm{rem},P}(\zeta_P) + \varrho'(\|\zeta_P\|_{H_P^\nu}^2)\|\zeta_P\|_{H_P^\nu}^2. \tag{3.3}$$

Let us now use Lemma 3.2, which asserts

$$\varrho(\|\zeta_P\|_{H_P^\nu}^2) + \mathcal{E}_P(\zeta_P) < q(1 - mq^{\frac{2}{3}}) \leq q. \tag{3.4}$$

Remark that one has

$$\mathcal{E}_P(\zeta_P) \geq \int_{-P/2}^{P/2} \gamma\frac{\zeta_P^2}{1-\zeta_P} + \frac{\zeta_P^2}{\delta^{-1}+\zeta_P} \,\mathrm{d}x = q + \mathcal{O}(q^{1+\frac{\epsilon}{2\nu}}),$$

where $\epsilon = \nu - 1/2 > 0$ and we use in the last estimate that $\|\zeta_P\|_{L^\infty}^2 \lesssim \|\zeta_P\|_{L_P^2}^{2-\frac{1}{\nu}}\|\zeta_P\|_{H_P^\nu}^{\frac{1}{\nu}} = \mathcal{O}(q^{\frac{2\nu-1}{2\nu}})$, by the interpolation estimate Lemma 2.1(i) in the periodic-setting. Combining with (3.4) yields

$$\mathcal{E}_P(\zeta_P) = q + \mathcal{O}(q^{1+\frac{\epsilon}{2\nu}}) \quad \text{ and } \quad \varrho(\|\zeta_P\|_{H_P^\nu}^2) = \mathcal{O}(q^{1+\frac{\epsilon}{2\nu}}). \tag{3.5}$$

Using Lemma 2.5 on one hand and assumption (3.1) on the other hand, we deduce

$$\|\zeta_P\|_{H_P^{1-\theta}}^2 \lesssim \mathcal{E}_P(\zeta_P) = \mathcal{O}(q) \quad \text{ and } \quad \varrho'(\|\zeta_P\|_{H_P^\nu}^2) = \mathcal{O}(q^{1+\frac{\epsilon}{4\nu}}).$$

Finally, we can estimate $\mathcal{E}_{3,P}$, $\mathcal{E}^{(1)}_{\mathrm{rem},P}$ and $\mathcal{E}^{(2)}_{\mathrm{rem},P}$ through Lemma 2.8 and combining previous estimates into (3.3) yields

$$-\alpha_P\, q = q + \mathcal{O}(q^{1+\frac{\epsilon}{4\nu}}),$$

and the proof is complete. □

Lemma 3.4. *Let q_0, $q \in (0, q_0)$ and P_q be as in Lemma 3.3. There exists $M > 0$ such that one has*

$$\|\zeta\|_{H_P^\nu}^2 \leq Mq$$

uniformly over $q \in (0, q_0)$, $P \geq P_q$ and ζ in the set of minimizers of $\mathcal{E}_{P,\varrho}$ over $V_{P,q,2R}$.

Proof. It follows from the proof of Lemma 3.3 that for q_0 sufficiently small $\big\|\zeta_P\big\|_{H_P^{1-\theta}}^2 \lesssim q$ with $0 \leq \theta < 1$. Thus the result is proved if $\nu = 1-\theta$, and we focus below on the situation $\nu > 1-\theta$. In this case we obtain the desired estimate in a similar fashion after finite induction. Indeed, define $r_n = \min(\nu-(1-\theta), n(1-\theta))$, $n \in \mathbb{N}$, and assume that $\big\|\zeta_P\big\|_{H_P^{r_n}}^2 \lesssim q$. Note that this is satisfied for $n = 0$ by assumption. We will show below that

$$\big\|\zeta_P\big\|_{H_P^{1-\theta+r_n}}^2 \lesssim \big\|\zeta_P\big\|_{H_P^{r_n}}^2 \lesssim q. \tag{3.6}$$

Since $1-\theta > 0$, the desired result follows by finite induction.

Let us now prove (3.6). We test (3.2) against $\Lambda^{2r_n}\zeta_P$, and obtain

$$2\varrho'(\big\|\zeta_P\big\|_{H_P^\nu}^2)\langle\Lambda^{2\nu}\zeta_P, \Lambda^{2r_n}\zeta_P\rangle + \langle \mathrm{d}\mathcal{E}_P(\zeta_P), \Lambda^{2r_n}\zeta_P\rangle + 2\alpha_P(\gamma+\delta)\langle\zeta_P, \Lambda^{2r_n}\zeta_P\rangle = 0. \tag{3.7}$$

Here, the notation $\langle , \rangle$ represents the $H_P^{\nu-2(1-\theta)} - H_P^{-\nu+2(1-\theta)}$ duality bracket. We will use the same notation for $H_P^s - H_P^{-s}$, where the value of $s \in (-\nu, \nu]$ is clear from the context. Note that all the terms are well-defined, since $\zeta_P \in H_P^\nu$, and therefore by Lemma 2.3, $\mathrm{d}\mathcal{E}_P(\zeta_P) \in H_P^{\nu-2(1-\theta)}$. Moreover, by (3.2), if $\varrho'(\big\|\zeta_P\big\|_{H_P^\nu}^2) > 0$ then $\Lambda^{2\nu}\zeta_P \in H_P^{\nu-2(1-\theta)}$ as well. Finally, $\Lambda^{2r_n}\zeta_P \in H_P^{\nu-2r_n}$, and $r_n + 1 - \theta \leq \nu$ so that $\nu - 2r_n \geq -\nu + 2(1-\theta)$.

Now, using that $\varrho'(\big\|\zeta_P\big\|_{H_P^\nu}^2) \geq 0$, we get from (3.7) and Lemma 3.3 that

$$\gamma\langle \mathrm{d}\overline{\mathcal{E}}_P(\zeta_P), \Lambda^{2r_n}\zeta_P\rangle + \langle \mathrm{d}\underline{\mathcal{E}}_P(\zeta_P), \Lambda^{2r_n}\zeta_P\rangle \leq 2(-\alpha_P)(\gamma+\delta)\big\|\Lambda^{r_n}\zeta_P\big\|_{L_P^2}^2 \leq 3(\gamma+\delta)\big\|\zeta_P\big\|_{H_P^{r_n}}^2. \tag{3.8}$$

Consider the first contribution, namely

$$\begin{aligned}\langle \mathrm{d}\overline{\mathcal{E}}(\zeta_P), \Lambda^{2r_n}\zeta_P\rangle = \left\langle \frac{2\zeta_P - \zeta_P^2}{(1-\zeta_P)^2}, \Lambda^{2r_n}\zeta_P\right\rangle + \left\langle (1-\zeta_P)^2\left(\partial_x \mathsf{F}\left\{\frac{\zeta_P}{1-\zeta_P}\right\}\right)^2, \Lambda^{2r_n}\zeta_P\right\rangle \\ + \left\langle \frac{2}{3}(1-\zeta_P)^3\partial_x\mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\}, \partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\rangle.\end{aligned} \tag{3.9}$$

We estimate each term of (3.9), using that $(\gamma+\delta)\big\|\zeta_P\big\|_{L_P^2}^2 = q$, $\big\|\zeta_P\big\|_{H_P^\nu} < 2R$ and $\big\|\zeta_P\big\|_{L^\infty} < 1 - h_0$, recalling that Lemmas 2.1, 2.2, 2.3 and 2.4(i) are valid in the periodic setting.

The first term in (3.9) is estimated by the Cauchy–Schwarz inequality and Lemma 2.2:

$$\left|\left\langle \frac{2\zeta_P - \zeta_P^2}{(1-\zeta_P)^2}, \Lambda^{2r_n}\zeta_P\right\rangle\right| \leq \Big\|\frac{2\zeta_P - \zeta_P^2}{(1-\zeta_P)^2}\Big\|_{H_P^{r_n}}\big\|\zeta_P\big\|_{H_P^{r_n}} \lesssim \big\|\zeta_P\big\|_{H_P^{r_n}}^2.$$

Next we see, using Lemmas 2.3(ii), 2.4(i) and finally Lemma 2.1(ii),

$$\begin{aligned}&\left|\left\langle (1-\zeta_P)^2\left(\partial_x\mathsf{F}\left\{\frac{\zeta_P}{1-\zeta_P}\right\}\right)^2, \Lambda^{2r_n}\zeta_P\right\rangle\right| \\ &\quad\lesssim \Big\|\left(\partial_x\mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\}\right)^2\Big\|_{H_P^{\theta-1+r_n}}\big\|\zeta_P\big\|_{H_P^{1-\theta+r_n}} \\ &\quad\lesssim \Big\|\partial_x\mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\}\Big\|_{H_P^{r_n}}\Big\|\partial_x\mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\}\Big\|_{H_P^{\nu+\theta-1-\epsilon}}\big\|\zeta_P\big\|_{H_P^{1-\theta+r_n}} \\ &\quad\lesssim \big\|\zeta_P\big\|_{H_P^{1-\theta+r_n}}^2\big\|\zeta_P\big\|_{H_P^{\nu-\epsilon}} \lesssim q^{\frac{\epsilon}{2\nu}}\big\|\zeta_P\big\|_{H_P^{1-\theta+r_n}}^2,\end{aligned}$$

for any $0<\epsilon<\min(\nu+\theta-1,1-\theta,\nu-1/2)$. Finally, note that

$$\begin{aligned}
&\left\langle (1-\zeta_P)^3\partial_x \mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\rangle \\
&\quad= \langle \partial_x\mathsf{F}_1\{\zeta_P\},\partial_x\mathsf{F}_1\{\Lambda^{2r_n}\zeta_P\}\rangle \\
&\qquad+\underbrace{\left\langle (-3\zeta_P+3\zeta_P^2-\zeta_P^3)\partial_x \mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\rangle}_{I} \\
&\qquad+\underbrace{\left\langle \partial_x \mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}-\zeta_P\right\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\rangle}_{II} \\
&\qquad+\underbrace{\left\langle \partial_x \mathsf{F}_1\{\zeta_P\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}-\Lambda^{2r_n}\zeta_P\right\}\right\rangle}_{III}.
\end{aligned}$$

First we see that, by Lemma 2.4(i),

$$\|\zeta\|_{H_P^{r_n}}^2+\langle \partial_x\mathsf{F}_1\{\zeta_P\},\partial_x\mathsf{F}_1\{\Lambda^{2r_n}\zeta_P\}\rangle \gtrsim \|\zeta\|^2_{H_P^{1-\theta+r_n}}.$$

We estimate I proceeding as previously:

$$\begin{aligned}
\left|\left\langle \zeta_P\partial_x \mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\rangle\right| &\lesssim \left\|\partial_x\mathsf{F}_1\left\{\frac{\zeta_P}{1-\zeta_P}\right\}\right\|_{H_P^{r_n}}\left\|\zeta_P\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\|_{H_P^{-r_n}}\\
&\lesssim \|\zeta_P\|_{H_P^{1-\theta+r_n}}\|\zeta_P\|_{H_P^{\nu-\epsilon}}\left\|\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\|_{H_P^{-r_n}}\\
&\lesssim \|\zeta_P\|_{H_P^{\nu-\epsilon}}\|\zeta_P\|^2_{H_P^{1-\theta+r_n}}\lesssim q^{\frac{\epsilon}{2\nu}}\|\zeta_P\|^2_{H_P^{1-\theta+r_n}},
\end{aligned}$$

where we choose $0<\epsilon<\min\{\nu-1/2,1-\theta\}$. The remaining terms in I are of higher order and can be estimated in the same way. Next we estimate II:

$$\begin{aligned}
\left|\left\langle \partial_x \mathsf{F}_1\left\{\frac{\zeta_P^2}{1-\zeta_P}\right\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}\right\}\right\rangle\right| &\lesssim \|\zeta_P^2\|_{H_P^{1-\theta+r_n}}\|\Lambda^{2r_n}\zeta_P\|_{H_P^{1-\theta-r_n}}\\
&\lesssim q^{\frac{\nu-1/2}{2\nu}}\|\zeta_P\|^2_{H_P^{1-\theta+r_n}},
\end{aligned}$$

where we used Lemmas 2.3(i) and 2.1(i). Finally consider III: proceeding as above,

$$\begin{aligned}
\left|\left\langle \partial_x \mathsf{F}_1\{\zeta_P\},\partial_x\mathsf{F}_1\left\{\frac{\Lambda^{2r_n}\zeta_P}{(1-\zeta_P)^2}-\Lambda^{2r_n}\zeta_P\right\}\right\rangle\right| &\lesssim \|\zeta_P\|_{H_P^{1-\theta+r_n}}\left\|\frac{2\zeta_P-\zeta_P^2}{(1-\zeta_P)^2}\Lambda^{2r_n}\zeta_P\right\|_{H_P^{1-\theta-r_n}}\\
&\lesssim q^{\frac{\epsilon}{2\nu}}\|\zeta_P\|^2_{H_P^{1-\theta+r_n}},
\end{aligned}$$

with $0<\epsilon<\min(\nu-1/2,\nu-(1-\theta),1-\theta)$.

Collecting the previous estimates into (3.9) yields

$$\|\zeta\|^2_{H_P^{1-\theta+r_n}}\lesssim \langle \mathrm{d}\overline{\mathcal{E}}(\zeta_P),\Lambda^{2r_n}\zeta_P\rangle+\|\zeta_P\|^2_{H_P^{r_n}}+q^{\frac{\epsilon}{2\nu}}\|\zeta_P\|^2_{H_P^{1-\theta+r_n}}$$

with some $\epsilon>0$. It is clear that the same estimate holds for $\langle \mathrm{d}\underline{\mathcal{E}}(\zeta_P),\Lambda^{2r_n}\zeta_P\rangle$. Using these in (3.8) and choosing q sufficiently small immediately imply (3.6). This concludes the proof. □

We now collect the preceding results and deduce the existence of solutions of the non-penalized periodic problem.

Theorem 3.5. (Existence of periodic minimizers) *There exists $q_0 > 0$ such that for any $q \in (0, q_0)$, one can define $P_q > 0$ and the following holds. For each $P \geq P_q$, there exists $\zeta_P \in V_{P,q,R}$ such that*

$$\mathcal{E}_P(\zeta_P) = \inf_{\zeta \in V_{P,q,R}} \mathcal{E}_P(\zeta) \stackrel{\text{def}}{=} I_{P,q}$$

and the Euler–Lagrange equation holds with $\alpha_P \in (-3/2, -1/2)$:

$$\mathrm{d}\mathcal{E}_P(\zeta_P) + 2\alpha_P(\gamma + \delta)\zeta_P = 0. \tag{3.10}$$

Furthermore, there exists $M > 0$, independent of q, such that

$$\left\|\zeta_P\right\|^2_{H^\nu_P} \leq Mq$$

uniformly with respect to $P \geq P_q$.

Proof. From Lemma 3.4, any minimizer of $\mathcal{E}_{P,\varrho}$ over $V_{P,q,2R}$ satisfies, for q_0 sufficiently small and $P \geq P_q$ sufficiently large,

$$\left\|\zeta_P\right\|^2_{H^\nu_P} \leq Mq < R^2.$$

Thus the Euler–Lagrange equation (3.2) becomes (3.10), and the control on α_P is stated in Lemma 3.3. Moreover, since $\mathcal{E}_{P,\varrho} = \mathcal{E}_P$ over $V_{P,q,R}$, ζ_P minimizes $\mathcal{E}_P$ over $V_{P,q,R}$. The theorem is proved. □

Remark 3.6. If $\theta \in [0, 1/2)$ and $\nu = 1 - \theta$, then the functional $\mathcal{E}_P$ is coercive on $V_{P,q,R}$ by Lemma 2.5, and it isn't necessary to consider the penalized functional $\mathcal{E}_{P,\varrho}$ to construct periodic minimizers. Indeed, one can minimize $\mathcal{E}_P$ over $V_{P,q,R}$ directly, noting that any minimizing sequence satisfies (up to subsequences) $\sup_n \left\|\zeta_{P,n}\right\|^2_{H^\nu_P} \leq Mq < R^2$ if q_0 is sufficiently small.

4. The Real Line Problem

The construction of a minimizer for the real line problem (1.9), will follow from Lions' concentration-compactness principle. The main difficulty consists in excluding the "dichotomy" scenario. To this aim, we shall use a special minimizing sequence (satisfying the additional estimate $\left\|\zeta_n\right\|^2_{H^\nu} \lesssim q$) to show that the function $q \mapsto I_q$ is strictly subhomogeneous (see Proposition 4.2), which implies that it is also strictly subadditive (Corollary 4.3). This special subsequence is constructed from the solutions of the periodic problem, obtained in Theorem 3.5, with period $P_n \to \infty$.

4.1. A Special Minimizing Sequence

Theorem 4.1. (Special minimizing sequence for $\mathcal{E}$) *There exists $q_0 > 0$ such that for any $q \in (0, q_0)$, one can define constants $m, M > 0$ and a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ satisfying*

$$(\gamma + \delta)\left\|\zeta_n\right\|^2_{L^2} = q, \quad \left\|\zeta_n\right\|^2_{H^\nu} \leq Mq$$

and

$$\lim_{n\to\infty} \mathcal{E}(\zeta_n) = I_q \stackrel{\text{def}}{=} \inf_{\zeta \in V_{q,R}} \mathcal{E}(\zeta) < q(1 - mq^{\frac{2}{3}}).$$

Proof. The estimate on I_q was proved in Lemma 3.2; thus we only need to construct a minimizing sequence satisfying $\left\|\zeta_n\right\|^2_{H^\nu} \leq Mq$. If $\nu = 1 - \theta$, then any minimizing sequence satisfies this property as a consequence of Lemma 2.5, so we assume in the following that $\nu > 1 - \theta$. Let q_0 be sufficiently small so that Theorem 3.5 holds. By the construction of [24, p. 2918 and proof of Theorem 3.8], one obtains, for any P_n sufficiently large, $x_n \in \mathbb{R}$, $\widetilde{\zeta}_{P_n} \in H^\nu_{P_n}$ and $\zeta_n \in H^\nu(\mathbb{R})$ such that

$$\left\|\widetilde{\zeta}_{P_n} - \zeta_{P_n}(\cdot - x_n)\right\|_{L^2_{P_n}} \to 0 \quad (P_n \to \infty) \tag{4.1}$$

where ζ_{P_n} is defined by Theorem 3.5,

$$\operatorname{supp}\zeta_n \subset (-P_n/2 + P_n^{1/2}, P_n/2 - P_n^{1/2}) \quad \text{and} \quad \widetilde{\zeta}_{P_n} = \sum_{l\in\mathbb{Z}} \zeta_n(\cdot + lP_n). \tag{4.2}$$

Moreover, one has

$$\|\zeta_n\|_{L^2} = \|\widetilde{\zeta}_{P_n}\|_{L^2_P} = \|\zeta_{P_n}\|_{L^2_P} \tag{4.3}$$

and

$$\|\zeta_n\|_{H^\nu} \lesssim \|\widetilde{\zeta}_{P_n}\|_{H^\nu_{P_n}} \lesssim \|\zeta_{P_n}\|_{H^\nu_{P_n}} \tag{4.4}$$

uniformly with respect to P_n sufficiently large.

By (4.4) and Theorem 3.5, one has $\|\zeta_n\|^2_{H^\nu} \leq Mq < R^2$ provided that P_n is sufficiently large and q_0 is sufficiently small; and $\zeta_n \in V_{q,R}$ by (4.3). Thus there only remains to prove that ζ_n is a minimizing sequence.

Here again we may proceed as in [24, Lemma 3.3 and Theorem 3.8]. Using in particular Lemma 2.4(iii), we find that

$$\mathcal{E}_{P_n}(\widetilde{\zeta}_{P_n}) - \mathcal{E}(\zeta_n) \to 0 \quad (P_n \to \infty).$$

Now by Lemma 2.6 (which holds in the periodic setting and uniformly with respect to $P > 0$) with ν replaced by some $\nu' \in (1/2, \nu)$ and Lemma 2.1(ii) with (4.1) and (4.4), one has

$$\mathcal{E}_{P_n}(\widetilde{\zeta}_{P_n}) - I_{P_n,q} = \mathcal{E}_{P_n}(\widetilde{\zeta}_{P_n}) - \mathcal{E}_{P_n}(\zeta_{P_n}(\cdot - x_n)) \to 0 \quad (P_n \to \infty).$$

Thus we found that

$$I_q \leq \mathcal{E}(\zeta_n) = I_{P_n,q} + o(1) \quad (P_n \to \infty).$$

There remains to prove the converse inequality. For any $\epsilon > 0$, there exists $\zeta \in V_{q,R}$ such that $\mathcal{E}(\zeta) \leq I_q + \frac{\epsilon}{3}$. By the same argument as above, we construct by smoothly truncating and rescaling, $\check{\zeta} \in V_{q,R}$ such that $\operatorname{supp}\check{\zeta} \in (-P_\star, P_\star)$, and $\mathcal{E}(\check{\zeta}) \leq \mathcal{E}(\zeta) + \frac{\epsilon}{3}$. Then for $P_n \geq 2P_\star$, one has $\check{\zeta}_{P_n} = \sum_{j\in\mathbb{Z}} \check{\zeta}(\cdot + jP_n) \in V_{P,q,R}$ and, as above, $\mathcal{E}_{P_n}(\check{\zeta}_{P_n}) - \mathcal{E}(\check{\zeta}) \to 0$ as $P_n \to \infty$. Hence for P_n sufficiently large, we have

$$I_{P_n,q} \leq \mathcal{E}_{P_n}(\check{\zeta}_{P_n}) \leq I_q + \epsilon.$$

Thus we proved that $\mathcal{E}(\zeta_n) \to I_q$ as $P_n \to \infty$, which concludes the proof. □

The following proposition is essential to rule out the "dichotomy" scenario in Lions' concentration-compactness principle (see below).

Proposition 4.2. *There exists $q_0 > 0$ such that the map $q \mapsto I_q$ is strictly subhomogeneous for $q \in (0, q_0)$:*

$$I_{aq} < aI_q \quad \textit{whenever} \quad 0 < q < aq < q_0.$$

Proof. Let us consider ζ_n the special minimizing sequence defined in Theorem 4.1. We first fix $a_0 > 1$, and restrict $q_0 > 0$ if necessary, so that for any $a \in (1, a_0]$ and $q \in (0, q_0)$ such that $aq < q_0$, one has $\|a^{\frac{1}{2}}\zeta_n\|^2_{H^\nu} \leq Maq \leq Mq_0 < R^2$. Thus we have, by definition of I_{aq} and Lemma 2.8,

$$I_{aq} \leq \mathcal{E}(a^{\frac{1}{2}}\zeta_n) = a\mathcal{E}(\zeta_n) + (a^{\frac{3}{2}} - a)\mathcal{E}_3(\zeta_n) + \mathcal{E}^{(1)}_{\rm rem}(a^{\frac{1}{2}}\zeta_n) - a\mathcal{E}^{(1)}_{\rm rem}(\zeta_n).$$

By Theorem 4.1 and Lemma 2.8 one has, for $q \in (0, q_0)$ with q_0 sufficiently small,

$$\limsup_{n\to\infty} \mathcal{E}_3(\zeta_n) = \limsup_{n\to\infty} \big(\mathcal{E}(\zeta_n) - \mathcal{E}_2(\zeta_n) - \mathcal{E}^{(1)}_{\rm rem}(\zeta_n)\big) \leq -\frac{1}{2}mq^{\frac{5}{3}}.$$

Thus we find, using again Theorem 4.1,

$$I_{aq} \leq aI_q - \frac{m}{2}(a^{\frac{3}{2}} - a)q^{\frac{5}{3}} + \limsup_{n\to\infty} \big(\mathcal{E}^{(1)}_{\rm rem}(a^{\frac{1}{2}}\zeta_n) - a\mathcal{E}^{(1)}_{\rm rem}(\zeta_n)\big). \tag{4.5}$$

We now estimate the last contribution, treating separately $\overline{\mathcal{E}}^{(1)}_{\rm rem}$ and $\underline{\mathcal{E}}^{(1)}_{\rm rem}$. Consider $\overline{\mathcal{E}}^{(1)}_{\rm rem}$ for instance. We develop each contribution in $\overline{\mathcal{E}}^{(1)}_{\rm rem}(a^{\frac{1}{2}}\zeta_n)$ using Neumann series in powers of $a^{\frac{1}{2}}\zeta_n$. The series are absolutely convergent provided q_0 is sufficiently small, and start at index $k = 4$. We now subtract the contributions of $a\overline{\mathcal{E}}^{(1)}_{\rm rem}(\zeta_n)$ and by the triangle and Cauchy–Schwarz inequalities,

$$|\overline{\mathcal{E}}^{(1)}_{\rm rem}(a^{\frac12}\zeta_n) - a\overline{\mathcal{E}}^{(1)}_{\rm rem}(\zeta_n)| \leq \sum_{k\geq 4}(a^{\frac{k}{2}} - a)\big\|\zeta_n\big\|^{k-2}_{L^\infty}\big\|\zeta_n\big\|^2_{L^2}$$
$$+ \sum_{k_1+k_2+k_3\geq 4} |c_{k_1,k_2,k_3}|(a^{\frac{k_1+k_2+k_3}{2}} - a)\big\|\zeta_n\big\|^{k_1}_{L^\infty}\big\|\partial_x\mathsf{F}_1\{\zeta_n^{k_2}\}\big\|_{L^2}\big\|\partial_x\mathsf{F}_1\{\zeta_n^{k_3}\}\big\|_{L^2}.$$

Using that $|a^{\frac{k}{2}} - a| \leq (a^{\frac32} - a)(k-2)a^{\frac{k-3}{2}}$, Lemma 2.4(i), that H^ν is a Banach algebra as well as the continuous embedding $H^\nu \subset L^\infty$, we find that one can restrict $q_0 > 0$ such that the above series is convergent and yields

$$|\overline{\mathcal{E}}^{(1)}_{\rm rem}(a^{\frac12}\zeta_n) - a\overline{\mathcal{E}}^{(1)}_{\rm rem}(\zeta_n)| \leq C(a_0)(a^{\frac32} - a)q^2,$$

uniformly over $q \in (0, q_0)$ and $a \in (1, a_0]$ such that $aq < q_0$. Plugging this estimate and the corresponding one for $\underline{\mathcal{E}}^{(1)}_{\rm rem}$ in (4.5) and restricting q_0 if necessary, we deduce

$$I_{aq} < aI_q \text{ for } 0 < q < aq < q_0,\ a \in (1, a_0].$$

Consider now the general case: $a \in (1, a_0^p]$ for an integer $p \geq 2$. Then $a^{\frac1p} \in (1, a_0]$ and so

$$I_{aq} = I_{a^{\frac1p}a^{\frac{p-1}{p}}q} < a^{\frac1p}I_{a^{\frac{p-1}{p}}q} = a^{\frac1p}I_{a^{\frac1p}a^{\frac{p-2}{p}}q} < a^{\frac2p}I_{a^{\frac{p-2}{p}}q} < \ldots < aI_q.$$

The result is proved. □

By a standard argument, Proposition 4.2 induces the subadditivity of the map $q \mapsto I_q$.

Corollary 4.3. *There exists $q_0 > 0$ such that the map $q \mapsto I_q$ is strictly subadditive for $q \in (0, q_0)$:*

$$I_{q_1+q_2} < I_{q_1} + I_{q_2} \quad \textit{whenever} \quad 0 < q_1 < q_1 + q_2 < q_0.$$

4.2. Concentration-Compactness: Proof of Theorem 1.6

We now prove Theorem 1.6. Let us first recall Lions' concentration-compactness principle [34].

Theorem 4.4. (Concentration-compactness) *Any sequence $\{e_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ of non-negative functions such that*

$$\lim_{n\to\infty}\int_{\mathbb{R}} e_n \,\mathrm{d}x = I > 0$$

admits a subsequence, denoted again $\{e_n\}_{n\in\mathbb{N}}$, for which one of the following phenomena occurs.

- *(Vanishing) For each $r > 0$, one has*
$$\lim_{n\to\infty}\left(\sup_{x\in\mathbb{R}}\int_{x-r}^{x+r} e_n \,\mathrm{d}x\right) = 0.$$
- *(Dichotomy) There are real sequences $\{x_n\}_{n\in\mathbb{N}}, \{M_n\}_{n\in\mathbb{N}}, \{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $I^* \in (0, I)$ such that $M_n, N_n \to \infty$, $M_n/N_n \to 0$, and*
$$\int_{x_n-M_n}^{x_n+M_n} e_n \,\mathrm{d}x \to I^* \quad \textit{and} \quad \int_{x_n-N_n}^{x_n+N_n} e_n \,\mathrm{d}x \to I^*$$
as $n \to \infty$.
- *(Concentration) There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with the property that for each $\epsilon > 0$, there exists $r > 0$ with*
$$\int_{x_n-r}^{x_n+r} e_n \,\mathrm{d}x \geq I - \epsilon$$
for all $n \in \mathbb{N}$.

We shall apply Theorem 4.4 to

$$e_n = \gamma\left(\frac{\zeta_n^2}{1-\zeta_n} + \frac{1}{3}(1-\zeta_n)^3\big(\partial_x \mathsf{F}_1\{\frac{\zeta_n}{1-\zeta_n}\}\big)^2\right) + \frac{\zeta_n^2}{\delta^{-1}+\zeta_n} + \frac{1}{3}(\delta^{-1}+\zeta_n)^3\big(\partial_x \mathsf{F}_2\{\frac{\zeta_n}{\delta^{-1}+\zeta_n}\}\big)^2,$$

where ζ_n is a minimizing sequence of $\mathcal{E}$ over $V_{q,R}$ with $\sup_n\|\zeta_n\|_{H^\nu}^2 < R^2$. Such a sequence is known to exist provided that $q \in (0, q_0)$ is sufficiently small, by Theorem 4.1 (and any minimizing sequence is valid when $\nu = 1-\theta$, by Lemma 2.5; see Remark 3.6). The choice of density is inspired by the recent paper [3], and allows (contrarily to the more evident choice $e_n = \zeta_n^2$) to show, when $\nu = 1-\theta$, that the constructed limit satisfies $\mathcal{E}(\eta) = I_q$ and is therefore a solution to the constrained minimization problem (1.9). Notice that

$$\int_{\mathbb{R}} e_n \, \mathrm{d}x = \mathcal{E}(\zeta_n) \to I_q \quad (n \to \infty)$$

and that there exists a constant C such that, for any interval $J \subseteq \mathbb{R}$,

$$\|\zeta_n\|_{L^2(J)}^2 = \int_J |\zeta_n|^2 \, \mathrm{d}x \le C \int_J e_n \, \mathrm{d}x. \tag{4.6}$$

We exclude the two first scenarios in Lemmas 4.5 and 4.6 below. Thus the concentration scenario holds and, using (4.6), we find that there exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there exists $r > 0$ with

$$\|\eta_n\|_{L^2(|x|>r)} < \epsilon,$$

where $\{\eta_n\}_{n\in\mathbb{N}} \stackrel{\text{def}}{=} \{\zeta_n(\cdot + x_n)\}_{n\in\mathbb{N}}$. Since $\sup_{n\in\mathbb{N}}\|\eta_n\|_{H^\nu(\mathbb{R})} < R$, there exists $\eta \in H^\nu(\mathbb{R})$ satisfying $\|\eta\|_{H^\nu(\mathbb{R})} < R$ and $\eta_n \rightharpoonup \eta$ weakly in $H^\nu(\mathbb{R})$ (up to the extraction of a subsequence). By compact embedding [4, Corollary 2.96] and Cantor's diagonal extraction process, one can extract a subsequence, still denoted η_n, such that $\|\eta_n - \eta\|_{L^2} \to 0$; and by interpolation $\|\eta_n - \eta\|_{H^s} \to 0$ for any $s \in [0, \nu)$. In particular $(\gamma + \delta)\|\eta\|_{L^2}^2 = q$, and $\|\eta\|_{H^\nu} \le \sup_n\|\zeta_n\|_{H^\nu} < R$, thus $\eta \in V_{q,R}$. If $\nu > 1 - \theta$, we deduce $\mathcal{E}(\eta_n) \to \mathcal{E}(\eta)$ as $n \to \infty$ by Lemma 2.6. If on the other hand $\nu = 1-\theta$ we use the weak lower semi-continuity argument to deduce that $I_q \le \mathcal{E}(\eta) \le \lim_{n\to\infty}\mathcal{E}(\eta_n) = I_q$. In either case we have that $\mathcal{E}(\eta) = I_q$.

The constructed function $\eta \in H^\nu(\mathbb{R})$ is therefore a solution to the constrained minimization problem (1.9). In particular, it solves the Euler–Lagrange equation (1.10) with $\alpha < 0$ provided that $q \in (0, q_0)$ is sufficiently small (proceeding as in Lemma 3.3), and therefore satisfies (1.8) with $c^2 = (-\alpha)^{-1} > 0$.

This proves the first item of Theorem 1.6, as well as the second item—except for the strong convergence in $H^\nu(\mathbb{R})$ when $\nu = 1 - \theta > 1/2$. This result follows from the fact that weak convergence together with convergence of the norm implies strong convergence in a Hilbert space (applied to $(\gamma^{1/2}(1-\zeta_n)^{3/2}\big(\partial_x\mathsf{F}_1\{\frac{\zeta_n}{1-\zeta_n}\}\big), (\delta^{-1}+\zeta_n)^{3/2}\big(\partial_x\mathsf{F}_2\{\frac{\zeta_n}{\delta^{-1}+\zeta_n}\}\big)) \in (L^2(\mathbb{R}))^2$).

There remains to prove the estimates of the third item. Proceeding as in Lemma 3.4, we find that $\|\zeta\|_{H^\nu}^2 \le Mq$, uniformly over the minimizers of $\mathcal{E}$ over $V_{q,R}$. Moreover, by Lemma 2.8, one has

$$\begin{aligned}
-\alpha\, q = -\alpha(\gamma+\delta)\|\zeta\|_{L^2}^2 = \frac{1}{2}\langle \mathrm{d}\mathcal{E}(\zeta), \zeta\rangle &= \mathcal{E}_2(\zeta) + \frac{3}{2}\mathcal{E}_3(\zeta) + \frac{1}{2}\mathcal{E}_{\mathrm{rem}}^{(2)}(\zeta) \\
&= \frac{3}{2}\mathcal{E}(\zeta) - \frac{1}{2}\mathcal{E}_2(\zeta) - \frac{3}{2}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta) + \frac{1}{2}\mathcal{E}_{\mathrm{rem}}^{(2)}(\zeta) \\
&< \frac{3}{2}q\left(1 - mq^{\frac{2}{3}}\right) - \frac{1}{2}q + \mathcal{O}(q^2).
\end{aligned}$$

where we used Lemmas 2.8 and 3.2 in the last inequality. Theorem 1.6 is proved.

Lemma 4.5. (Excluding "vanishing") *No subsequence of* $\{e_n\}_{n\in\mathbb{N}}$ *has the "vanishing" property.*

Proof. By Lemmas 2.8 and 3.2, one has for n sufficiently large

$$q(1-mq^{\frac{2}{3}}) > \mathcal{E}(\zeta_n) = \mathcal{E}_2(\zeta_n) + \mathcal{E}_3(\zeta_n) + \mathcal{E}^{(1)}_{\text{rem}}(\zeta_n) \\ \geq q + \mathcal{E}_3(\zeta_n) + \mathcal{E}^{(1)}_{\text{rem}}(\zeta_n)$$

and hence

$$mq^{\frac{5}{3}} \leq |\mathcal{E}_3(\zeta_n)| + |\mathcal{E}^{(1)}_{\text{rem}}(\zeta_n)| \lesssim \big\|\zeta_n\big\|_{L^\infty}.$$

On the other hand, one has

$$\big\|\zeta_n\big\|_{L^\infty((x-\frac{1}{2},x+\frac{1}{2}))} \leq \big\|\varphi_x\zeta_n\big\|_{L^\infty(\mathbb{R})} \leq \big\|\varphi_x\zeta_n\big\|_{L^2(\mathbb{R})}^{1-\frac{1}{2\nu}}\big\|\varphi_x\zeta_n\big\|_{H^\nu(\mathbb{R})}^{\frac{1}{2\nu}} \leq C\big\|\zeta_n\big\|_{L^2((x-1,x+1))}^{1-\frac{1}{2\nu}}\big\|\zeta_n\big\|_{H^\nu(\mathbb{R})}^{\frac{1}{2\nu}},$$

where $\varphi_x = \varphi(\cdot - x)$ with φ a smooth function such that $\varphi = 1$ for $|x| \leq 1/2$, $\varphi = 0$ for $|x| \geq 1$, and $0 \leq \varphi \leq 1$ otherwise; and using Lemmas 2.1(i) and 2.3(ii). Since C is independent of $x \in \mathbb{R}$, this shows that

$$\big\|\zeta_n\big\|_{L^\infty} \leq CR^{\frac{1}{2\nu}} \sup_{x\in\mathbb{R}} \big\|\zeta_n\big\|_{L^2((x-1,x+1))}^{1-\frac{1}{2\nu}}.$$

Hence, using (4.6), it follows that "vanishing" cannot occur. □

Lemma 4.6. (Excluding "dichotomy") *No subsequence of $\{e_n\}_{n\in\mathbb{N}}$ has the "dichotomy" property.*

Proof. We denote by $\chi \in \mathcal{C}^\infty(\mathbb{R}^+)$ a non-increasing function with

$$\chi(r) = 1 \text{ if } 0 \leq r \leq 1 \quad \text{ and } \quad \chi(r) = 0 \text{ if } r \geq 2, \tag{4.7}$$

and such that

$$\chi = \chi_1^2, \quad 1-\chi = \chi_2^2$$

where χ_1 and χ_2 are smooth. For instance, set $\chi(r) = 1-(1-\widetilde{\chi}^2(r))^2$ with $\widetilde{\chi} \in \mathcal{C}^\infty(\mathbb{R}^+)$ non-increasing and satisfying (4.7). Define $\eta_n = \zeta_n(\cdot + x_n)$, and

$$\eta_n^{(1)}(x) = \eta_n(x)\chi(|x|/M_n) \quad \text{ and } \quad \eta_n^{(2)}(x) = \eta_n(x)\Big(1-\chi(2|x|/N_n)\Big),$$

noting that

$$\text{supp}(\eta_n^{(1)}) \subset [-2M_n, 2M_n] \quad \text{ and } \quad \text{supp}(\eta_n^{(2)}) \subset \mathbb{R}\setminus[-N_n/2, N_n/2].$$

After possibly extracting a subsequence, we can assume that

$$\big\|\eta_n^{(1)}\big\|_{L^2}^2 \to \frac{q^*}{\gamma+\delta} \quad (n\to\infty) \tag{4.8}$$

with $q^* \in [0,q]$. By (4.6) and the assumption of the dichotomy scenario, we have

$$\big\|\eta_n\big\|_{L^2(M_n<|x|<N_n)}^2 \leq C\int_{M_n<|x-x_n|<N_n} e_n \mathrm{d}x \to 0.$$

Hence, proceeding as in [24, Proposition 5.4], we find that

$$\big\|\eta_n^{(2)}\big\|_{L^2}^2 \to \frac{q-q^*}{\gamma+\delta} \quad (n\to\infty). \tag{4.9}$$

We claim that $\mathcal{E}(\eta_n^{(1)}) \to I^*$. To show this, note that

$$\overline{\mathcal{E}}(\eta_n^{(1)}) = \int_{\mathbb{R}} \frac{(\eta_n^{(1)})^2}{1-\eta_n^{(1)}} + \frac{1}{3}(1-\eta_n^{(1)})^3\big(\partial_x \mathsf{F}_1\{\frac{\eta_n^{(1)}}{1-\eta_n^{(1)}}\}\big)^2 \mathrm{d}x,$$

where

$$\left|\int_{\mathbb{R}} \frac{(\eta_n^{(1)})^2}{1-\eta_n^{(1)}}\mathrm{d}x - \int_{|x|\leq M_n} \frac{\eta_n^2}{1-\eta_n}\mathrm{d}x\right| \lesssim \int_{M_n\leq|x|\leq N_n} \eta_n^2 \mathrm{d}x \to 0.$$

The other contribution is more involved due to the nonlocal operator $\partial_x \mathsf{F}_1$. However, using Lemmas 2.3(i) and 2.1(i), and the fact that $\|\eta_n^{(1)}\|_{H^\nu} \lesssim \|\eta_n\|_{H^\nu} \leq R$, we find that

$$\left|\int_{\mathbb{R}} (1-\eta_n^{(1)})^3 \big(\partial_x \mathsf{F}_1\{\frac{\eta_n^{(1)}}{1-\eta_n^{(1)}}\}\big)^2 - (1-\eta_n^{(1)})^3 \big(\partial_x \mathsf{F}_1\{\frac{\eta_n^{(1)}}{1-\eta_n}\}\big)^2 \mathrm{d}x\right| \lesssim \|\eta_n\|_{L^2(M_n\leq|x|\leq N_n)}^{1-\frac{1}{2\nu}} \to 0,$$

and

$$\left|\int_{\mathbb{R}} (1-\eta_n^{(1)})^3 \chi^2(|\cdot|/M_n) \big(\partial_x \mathsf{F}_1\{\frac{\eta_n}{1-\eta_n}\}\big)^2 \mathrm{d}x - \int_{|x|\leq M_n} (1-\eta_n)^3 \big(\partial_x \mathsf{F}_1\{\frac{\eta_n}{1-\eta_n}\}\big)^2 \mathrm{d}x\right| \to 0.$$

Finally, by Lemma 2.4(ii), one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}} (1-\eta_n^{(1)})^3 \big(\partial_x \mathsf{F}_1\{\frac{\eta_n^{(1)}}{1-\eta_n}\}\big)^2 - (1-\eta_n^{(1)})^3 \chi^2(|\cdot|/M_n) \big(\partial_x \mathsf{F}_1\{\frac{\eta_n}{1-\eta_n}\}\big)^2 \mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}} (1-\eta_n^{(1)})^3 \Big(\big(\partial_x \mathsf{F}_1\{\frac{\eta_n^{(1)}}{1-\eta_n}\}\big) + \chi(|\cdot|/M_n)\big(\partial_x \mathsf{F}_1\{\frac{\eta_n}{1-\eta_n}\}\big)\Big) [\partial_x \mathsf{F}_1, \chi(|\cdot|/M_n)]\big(\frac{\eta_n}{1-\eta_n}\big) \mathrm{d}x\right| \\
&\lesssim M_n^{-1} \|\eta_n\|_{H^\nu}^2 \to 0.
\end{aligned}$$

Altogether, and since an analogous argument evidently holds for $\underline{\mathcal{E}}$, we find

$$\mathcal{E}(\eta_n^{(1)}) = \int_{|x-x_n|\leq M_n} e_n \mathrm{d}x + o(1) \to I^*$$

and by similar reasoning one finds that

$$\mathcal{E}(\eta_n^{(2)}) = \int_{|x-x_n|\geq N_n} e_n \mathrm{d}x + o(1) \to I_q - I^*.$$

We next claim that $q^* > 0$. Indeed, if $q^* = 0$, we set

$$\widetilde{\eta}_n^{(2)} \stackrel{\text{def}}{=} c_n \eta_n^{(2)}, \qquad c_n \stackrel{\text{def}}{=} \frac{q^{\frac{1}{2}}}{(\gamma+\delta)^{\frac{1}{2}} \|\eta_n^{(2)}\|_{L^2}}.$$

By (4.9) and since $q^* = 0$, one has $c_n \to 1$. Thus by Lemma 2.6 and since $\limsup_{n\to\infty} \|\widetilde{\eta}_n^{(2)}\|_{H^\nu} < R$,

$$|\mathcal{E}(\widetilde{\eta}_n^{(2)}) - \mathcal{E}(\eta_n^{(2)})| \lesssim \|\widetilde{\eta}_n^{(2)} - \eta_n^{(2)}\|_{H^\nu} \to 0$$

resulting in the contradiction $I_q \leq \mathcal{E}(\widetilde{\eta}_n^{(2)}) \to I_q - I^* < I_q$ as $n \to \infty$. We obtain a similar contradiction involving $\eta_n^{(1)}$ and (4.8) if we assume that $q^* = q$. Hence, $0 < q^* < q$, and we rescale

$$\widetilde{\eta}_n^{(1)} \stackrel{\text{def}}{=} \frac{(q^*)^{\frac{1}{2}}}{(\gamma+\delta)^{\frac{1}{2}} \|\eta_n^{(1)}\|_{L^2}} \eta_n^{(1)} \quad \text{and} \quad \widetilde{\eta}_n^{(2)} \stackrel{\text{def}}{=} \frac{(q-q^*)^{\frac{1}{2}}}{(\gamma+\delta)^{\frac{1}{2}} \|\eta_n^{(2)}\|_{L^2}} \eta_n^{(2)},$$

so that $(\gamma+\delta)\|\widetilde{\eta}_n^{(1)}\|_{L^2}^2 = q$ and $(\gamma+\delta)\|\widetilde{\eta}_n^{(2)}\|_{L^2}^2 = q - q_*$ for any $n \in \mathbb{N}$. One easily checks that

$$\limsup_{n\to\infty} \|\widetilde{\eta}_n^{(1)}\|_{H^\nu} < R, \quad \limsup_{n\to\infty} \|\widetilde{\eta}_n^{(2)}\|_{H^\nu} < R$$

and that

$$\lim_{n\to\infty} (\mathcal{E}(\widetilde{\eta}_n^{(1)}) - \mathcal{E}(\eta_n^{(1)})) = \lim_{n\to\infty} (\mathcal{E}(\widetilde{\eta}_n^{(2)}) - \mathcal{E}(\eta_n^{(2)})) = 0.$$

Thus we arrive at the following contradiction to Corollary 4.3:

$$I_q < I_{q^*} + I_{q-q^*} \leq \lim_{n\to\infty} (\mathcal{E}(\widetilde{\eta}_n^{(1)}) + \mathcal{E}(\widetilde{\eta}_n^{(2)})) = I^* + I_q - I^* = I_q.$$

This concludes the proof of Lemma 4.6. □

5. Long-Wave Asymptotics

In this section we prove that the solutions of (1.8) obtained in Theorem 1.6 are approximated by solutions of the corresponding KdV equation in the long-wave regime, *i.e.* letting $q \to 0$ in the constrained minimization problem (1.9). Indeed, if we introduce the scaling

$$\zeta(x) = S_{\rm KdV}(\xi)(x) \stackrel{\rm def}{=} q^{\frac{2}{3}}\xi(q^{\frac{1}{3}}x) \tag{5.1}$$

in (1.10) and denote $\alpha + 1 = \alpha_0 q^{\frac{2}{3}}$, then we find that the leading order part of the equation as $q \to 0$ is

$$\alpha_0(\gamma+\delta)\xi + \frac{3(\gamma-\delta^2)\xi^2}{2} - \frac{(\gamma+\delta^{-1})}{3}\partial_x^2\xi = 0. \tag{5.2}$$

Recall (see *e.g.* [2]) that $\xi \in L^2(\mathbb{R})$ satisfying (5.2) uniquely defines (up to spatial translation) a solitary-wave solution of the KdV equation, with explicit formula

$$\xi_{\rm KdV}(x) = \frac{\alpha_0(\gamma+\delta)}{\delta^2-\gamma}\,{\rm sech}^2\left(\frac{1}{2}\sqrt{\frac{3\alpha_0(\gamma+\delta)}{\gamma+\delta^{-1}}}x\right). \tag{5.3}$$

Equation (5.2) can also be obtained as the Euler–Lagrange equation associated with the minimizer of the scalar functional $\mathcal{E}_{\rm KdV}$ (consistently with Lemma 2.7)

$$\mathcal{E}_{\rm KdV}(\xi) = \int_{\mathbb{R}} (\gamma-\delta^2)\xi^3 + \frac{(\gamma+\delta^{-1})}{3}(\partial_x\xi)^2 \, {\rm d}x,$$

over the set

$$U_1 \stackrel{\rm def}{=} \{\xi \in H^1(\mathbb{R}) \ : \ (\gamma+\delta)\big\|\xi\big\|_{L^2}^2 = 1\}.$$

Indeed, any minimizer satisfies the Euler–Lagrange equation

$${\rm d}\mathcal{E}_{\rm KdV}(\xi) + 2(\gamma+\delta)\alpha_0\xi = 0, \tag{5.4}$$

which is (5.2) with α_0 the Lagrange multiplier. Testing the constraint $(\gamma+\delta)\big\|\xi\big\|_{L^2}^2 = 1$ with the above explicit formula, we find that

$$(\gamma+\delta)\alpha_0 = \frac{3}{4}\left(\frac{(\delta^2-\gamma)^4}{(\gamma+\delta)(\gamma+\delta^{-1})}\right)^{\frac{1}{3}}. \tag{5.5}$$

Additional computations show that

$$I_{\rm KdV} = \inf\{\mathcal{E}_{\rm KdV}(\xi) \ : \ \xi \in U_1\} = \mathcal{E}_{\rm KdV}(\xi_{\rm KdV}) = -\frac{3}{5}\alpha_0.$$

We aim at proving that the variational characterization of (5.2), and therefore its explicit solutions, approximate (after suitable rescaling) the corresponding one of (1.8), namely (1.9), in the limit $q \to 0$.

5.1. Refined Estimates

We start by establishing estimates on $\zeta \in D_{q,R}$ the set of minimizers of $\mathcal{E}$ over $V_{q,R}$, as provided by Theorem 1.6. Here and below, we rely on extra assumptions on the Fourier multipliers, which are assumed to be *strongly admissible*, in the sense of Definition 1.2.

Lemma 5.1. *There exists $q_0 > 0$ such that $\zeta \in H^s$ for any $s \geq 0$, and there exists $M_s > 0$ such that*

$$\big\|\zeta\big\|_{H^s}^2 \leq M_s\, q$$

uniformly for $q \in (0, q_0)$ and $\zeta \in D_{q,R}$.

Proof. Once the regularity property $\zeta \in H^s$ has been established, the corresponding estimate is obtained as in the proof of Lemma 3.4, thus we focus only on the regularity issue. This follows from the Euler–Lagrange equation (1.10) and elliptic estimates. However, the ellipticity property is not straightforward to ascertain when $\gamma \neq 0$, and we will make use of paradifferential calculus. These tools are recalled in the Appendix.

By assumption, one has $\zeta \in H^\nu$ with $\nu > 1/2$ and $\nu \geq 1-\theta > 0$. We fix $\epsilon \in (0, \nu - 1/2)$ and $r = \min(1-\theta, \nu - 1/2 - \epsilon) > 0$. We show below that $\zeta \in H^\nu$ satisfying (1.10) yields $\zeta \in H^{\nu+r}$, and the argument can be bootstrapped to obtain arbitrarily high regularity, $\zeta \in H^s$, $s \geq 0$.

First we write (1.10) as the equality, valid in $H^{-\nu}$,

$$\begin{aligned}
&\frac{2}{3} h_2^{-2} \partial_x \mathsf{F}_2 \big\{ h_2^3 \partial_x \mathsf{F}_2 \{ h_2^{-1} \zeta \} \big\} + \frac{2\gamma}{3} h_1^{-2} \partial_x \mathsf{F}_1 \big\{ h_1^3 \partial_x \mathsf{F}_1 \{ h_1^{-1} \zeta \} \big\} \\
&\quad = 2\alpha(\gamma+\delta)\zeta + 2\frac{h_1 + \gamma h_2}{h_1 h_2}\zeta - \frac{h_1^2 - \gamma h_2^2}{h_1^2 h_2^2}|\zeta|^2 + \big(h_2 \partial_x \mathsf{F}_2 \{ h_2^{-1} \zeta \} \big)^2 - \gamma \big(h_1 \partial_x \mathsf{F}_1 \{ h_1^{-1} \zeta \} \big)^2 \\
&\quad \stackrel{\text{def}}{=} R(\zeta)
\end{aligned} \tag{5.6}$$

denoting $h_1 = 1 - \zeta$, $h_2 = \delta^{-1} + \zeta$, and recalling $\alpha \in (-3/2, -1/2)$.

Using Lemmas 2.3(ii) and 2.4(i), one easily checks that $R(\zeta) \in H^{2(\nu-(1-\theta))-1/2-\epsilon}$ in the case $1/2 < \nu \leq 1/2 + (1-\theta)$, and $R(\zeta) \in H^{\nu-(1-\theta)}$ if $\nu > 1/2 + (1-\theta)$. In other words, we find

$$R(\zeta) \in H^{\nu - 2(1-\theta) + r}. \tag{5.7}$$

Above, we used that $\|\zeta\|_{L^\infty} < \min(1, \delta^{-1})$ and therefore $h_1(x)^n - 1 \in H^\nu$ and $h_2(x)^n - (\delta^{-1})^n \in H^\nu$ for any $n \in \mathbb{Z}$. This holds as well in the Hölder space $W^{r,\infty}$ since $r \in (0, \nu - 1/2)$. In particular, we have

$$\forall n \in \mathbb{Z}, \quad h_1(x)^n \in \Gamma_r^0 \qquad \text{and} \qquad \partial_x \mathsf{F}_i \in \Gamma_r^{1-\theta},$$

recalling Definition A.1.

Using Lemmas A.3, A.4, A.5 A.6 as well as Lemmas 2.3 and 2.4, one easily checks that

$$h_1^{-2} \partial_x \mathsf{F}_1 \big\{ h_1^3 \partial_x \mathsf{F}_1 \{ h_1^{-1} \zeta \} \big\} - T_{-(k\mathsf{F}_1(k))^2 h_1^{-1}} \zeta \in H^{\nu - 2(1-\theta) + r}. \tag{5.8}$$

By (5.7), (5.8) and the corresponding estimate for the second contribution in the left-hand side of (5.6), one finds

$$T_{\frac{2}{3} h_2^{-1} (\mathrm{i}k\mathsf{F}_2(k))^2 + \frac{2\gamma}{3} h_1^{-1} (\mathrm{i}k\mathsf{F}_1(k))^2} \zeta \in H^{\nu - 2(1-\theta) + r}.$$

Moreover, since $\zeta \in H^\nu$, one has $\frac{2}{3} h_2^{-1}(x) + \frac{2\gamma}{3} h_1^{-1}(x) \in \Gamma_r^0$ and therefore

$$T_{\frac{2}{3} h_2^{-1}(x) + \frac{2\gamma}{3} h_1^{-1}(x)} \zeta \in H^\nu \subset H^{\nu - 2(1-\theta) + r}.$$

Adding the two terms yields

$$T_{a(x,k)} \zeta \in H^{\nu - 2(1-\theta) + r} \tag{5.9}$$

with

$$a(x,k) \stackrel{\text{def}}{=} \frac{2}{3} h_2^{-1}(x) \big(1 + (k\mathsf{F}_2(k))^2 \big) + \frac{2\gamma}{3} h_1^{-1}(x) \big(1 + (k\mathsf{F}_1(k))^2 \big).$$

Notice that

$$a(x,k) \in \Gamma_r^{2(1-\theta)} \qquad \text{and} \qquad a(x,k)^{-1} \in \Gamma_r^{-2(1-\theta)}.$$

In particular, Lemma A.3 and (5.9) yield

$$T_{a(x,k)^{-1}} T_{a(x,k)} \zeta \in H^{\nu + r}.$$

Additionally, by Lemma A.4, we have

$$\zeta - T_{a(x,k)^{-1}} T_{a(x,k)} \zeta = T_{a(x,k)^{-1} a(x,k)} \zeta - T_{a(x,k)^{-1}} T_{a(x,k)} \zeta \in H^{\nu + r}.$$

Adding the two terms shows that $\zeta \in H^{\nu + r}$, which concludes the proof. □

Remark 5.2. In the one-layer situation, namely $\gamma = 0$, the use of paradifferential calculus is not necessary, and Lemma 5.1 can be obtained through a direct use of Lemmas 2.2 and 2.3. In particular, Lemma 5.1 and subsequent results hold for (non-necessarily strongly) admissible Fourier multipliers, in the sense of Definition 1.1.

The following lemma shows that the minimizers of $\mathcal{E}$ over $V_{q,R}$, as provided by Theorem 1.6, scale as (5.1).

Lemma 5.3. *There exists $q_0 > 0$ and $C > 0$ such that the estimates*

$$\big\|\zeta\big\|_{L^\infty} \leq Cq^{\frac{2}{3}}, \tag{5.10}$$

$$\big\|\partial_x\zeta\big\|_{L^2}^2 \leq Cq^{\frac{5}{3}}, \tag{5.11}$$

$$\big\|\partial_x^2\zeta\big\|_{L^2}^2 \leq Cq^{\frac{7}{3}} \tag{5.12}$$

hold uniformly for $q \in (0, q_0)$ and $\zeta \in D_{q,R}$, the set of minimizers of $\mathcal{E}$ over $V_{q,R}$.

Proof. Let ζ be minimizer over $V_{q,R}$. Since $2(\gamma+\delta)\alpha\zeta + \mathrm{d}\mathcal{E}(\zeta) = 0$, we get from Lemma 2.8 that

$$2\alpha(\gamma+\delta)\zeta + \mathrm{d}\mathcal{E}_2(\zeta) = 2\alpha(\gamma+\delta)\zeta + \mathrm{d}\mathcal{E}(\zeta) - \mathrm{d}\mathcal{E}_3(\zeta) - \mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta) = -\mathrm{d}\mathcal{E}_3(\zeta) - \mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta),$$

By using the estimate for α in Theorem 1.6, the above identity in frequency space yields

$$|\widehat{\zeta}(k)| \leq \frac{1}{2}\frac{\left|\mathcal{F}\Big(\mathrm{d}\mathcal{E}_3(\zeta) + \mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta)\Big)(k)\right|}{mq^{\frac{2}{3}} + \frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2 + \delta^{-1}(k\mathsf{F}_2(k))^2)}. \tag{5.13}$$

The estimates follow from (5.13) and a suitable decomposition into high- and low-frequency components. In order to estimate the right-hand-side, we heavily make use of Lemma 5.1: $\big\|\zeta\big\|_{H^n}^2 \lesssim q$ for all $n \in \mathbb{N}$. This will be used again throughout the proof without reference.

We first deduce from Lemma 2.3 that

$$\big\|\mathcal{F}\big(\mathrm{d}\mathcal{E}_3(\zeta)\big)\big\|_{L^1} \lesssim \big\|\mathrm{d}\mathcal{E}_3(\zeta)\big\|_{H^1} \lesssim q$$

and

$$\big\|\mathcal{F}\big(\mathrm{d}\mathcal{E}_3(\zeta)\big)\big\|_{L^\infty} \lesssim \big\|\zeta\big\|_{L^2}^2 + \big\|\partial_x\zeta\big\|_{L^2}^2 + \big\|\zeta\big\|_{L^2}\big\|\partial_x^2\zeta\big\|_{L^2} \lesssim q$$

and, similarly,

$$\big\|\mathcal{F}\big(\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta)\big)\big\|_{L^1} + \big\|\mathcal{F}\big(\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta)\big)\big\|_{L^\infty} \lesssim q^{\frac{3}{2}}.$$

By the definition of admissible Fourier multipliers in (1.1), there exists $c_0, k_0 > 0$ such that

$$\forall k \in \mathbb{R}\setminus[-k_0,k_0], \quad |k|\mathsf{F}_i(k) \geq c_0,\ i = 1,2.$$

We also assume that $\mathsf{F}(k) > 0$, and therefore there exists $c_0' > 0$ such that

$$\forall k \in [-k_0,k_0], \quad \mathsf{F}_i(k) \geq c_0',\ i = 1,2.$$

As a consequence, we have

$$\sup_{k\in\mathbb{R}\setminus[-k_0,k_0]} \frac{1}{mq^{\frac{2}{3}} + \frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2 + \delta^{-1}(k\mathsf{F}_2(k))^2)} \lesssim 1$$

and

$$\int_{-k_0}^{k_0} \frac{1}{mq^{\frac{2}{3}} + \frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2 + \delta^{-1}(k\mathsf{F}_2(k))^2)}\mathrm{d}k \lesssim q^{-\frac{1}{3}}.$$

Now, we have

$$\begin{aligned}\|\zeta\|_{L^\infty} &\le \frac{1}{\sqrt{2\pi}}\|\widehat{\zeta}\|_{L^1} \le \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\left|\mathcal{F}\Big(\mathrm{d}\mathcal{E}_3(\zeta)+\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta)\Big)(k)\right|}{mq^{\frac{2}{3}}+\frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2+\delta^{-1}(k\mathsf{F}_2(k))^2)}\mathrm{d}k\\ &\le \frac{1}{2\sqrt{2\pi}}\Big\|\mathcal{F}\Big(\mathrm{d}\mathcal{E}_3(\zeta)+\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta)\Big)(k)\Big\|_{L^\infty}\times\int_{-k_0}^{k_0}\frac{1}{mq^{\frac{2}{3}}+\frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2+\delta^{-1}(k\mathsf{F}_2(k))^2)}\mathrm{d}k\\ &\quad+\frac{1}{2\sqrt{2\pi}}\Big\|\mathcal{F}\Big(\mathrm{d}\mathcal{E}_3(\zeta)+\mathrm{d}\mathcal{E}_{\mathrm{rem}}(\zeta)\Big)(k)\Big\|_{L^1}\sup_{k\in\mathbb{R}\setminus[-k_0,k_0]}\frac{1}{mq^{\frac{2}{3}}+\frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2+\delta^{-1}(k\mathsf{F}_2(k))^2)},\end{aligned}$$

from which we immediately deduce the inequality (5.10).

Let us now turn to (5.11). By (5.13) we have

$$\|\partial_x\zeta\|_{L^2}^2 \le \frac{1}{4}\int_{\mathbb{R}}\frac{k^2|\mathcal{F}(\mathrm{d}\mathcal{E}_3(\zeta)+\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta))|^2}{(mq^{\frac{2}{3}}+\frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2+\delta^{-1}(k\mathsf{F}_2(k))^2))^2}\mathrm{d}k. \tag{5.14}$$

Notice also that $\|k\mathcal{F}(\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta))\|_{L^2}\lesssim q^{\frac{3}{2}}$ and, using (5.10),

$$\begin{aligned}\|k\mathcal{F}(\mathrm{d}\mathcal{E}_3(\zeta))\|_{L^2} &\lesssim \|\zeta\|_{L^\infty}\|\partial_x\zeta\|_{L^2}+\|\partial_x\zeta\|_{L^2}\|\partial_x^2\zeta\|_{H^1}+\|\zeta\|_{L^\infty}\|\partial_x^3\zeta\|_{L^2}\\ &\lesssim q^{\frac{7}{6}}+q^{\frac{1}{2}}\|\partial_x\zeta\|_{L^2}.\end{aligned}$$

Hence we have

$$\begin{aligned}\|\partial_x\zeta\|_{L^2}^2 &\lesssim \|\mathcal{F}(\mathrm{d}\mathcal{E}_3(\zeta)+\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta))(k)\|_{L^\infty}^2\times\int_{-k_0}^{k_0}\frac{k^2}{(mq^{\frac{2}{3}}+\frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2+\delta^{-1}(k\mathsf{F}_2(k))^2))^2}\,\mathrm{d}k\\ &\quad+\|k\mathcal{F}(\mathrm{d}\mathcal{E}_3(\zeta)+\mathrm{d}\mathcal{E}_{\mathrm{rem}}^{(1)}(\zeta))\|_{L^2}^2\times\sup_{k\in\mathbb{R}\setminus[-k_0,k_0]}\frac{1}{(mq^{\frac{2}{3}}+\frac{1}{3}(\gamma(k\mathsf{F}_1(k))^2+\delta^{-1}(k\mathsf{F}_2(k))^2))^2}\\ &\lesssim q^{\frac{5}{3}}+q\|\partial_x\zeta\|_{L^2}^2.\end{aligned}$$

This shows (5.11) for q_0 sufficiently small.

The proof of (5.12) is similar to the proof of (5.11) and is therefore omitted. □

5.2. Convergence Results: Proof of Theorem 1.7

We are now in position to relate the minimizers of $\mathcal{E}$ in $D_{q,R}$ with the corresponding solution of the KdV equation. We first compare I_{KdV} and I_q.

Lemma 5.4. *There exists $q_0>0$ such that the quantities I_q and I_{KdV} satisfy*

$$I_q = q+\mathcal{E}_{\mathrm{KdV}}(\zeta)+\mathcal{O}(q^2), \tag{5.15}$$

$$I_q = q+q^{\frac{5}{3}}I_{\mathrm{KdV}}+\mathcal{O}(q^2) = q-\frac{3}{5}\alpha_0 q^{\frac{5}{3}}+\mathcal{O}(q^2), \tag{5.16}$$

uniformly over minimizers of $\mathcal{E}$ in $V_{q,R}$ and $q\in(0,q_0)$.

Proof. Let ζ be a minimizer of $\mathcal{E}$ in $V_{q,R}$ and note that $\zeta\in H^2$ by Lemma 5.1. Using Lemmas 2.7 and 5.3, we obtain

$$I_q=\mathcal{E}(\zeta)=q+\mathcal{E}_{\mathrm{KdV}}(\zeta)+\mathcal{O}(q^2).$$

Introducing $\xi=S_{\mathrm{KdV}}^{-1}(\zeta)$, we find that $\xi\in U_1$ and $\mathcal{E}_{\mathrm{KdV}}(\zeta)=q^{\frac{5}{3}}\mathcal{E}_{\mathrm{KdV}}(\xi)\ge q^{\frac{5}{3}}I_{\mathrm{KdV}}$. Thus we found

$$I_q\ge q+q^{\frac{5}{3}}I_{\mathrm{KdV}}+\mathcal{O}(q^2).$$

Similarly, notice that $\widetilde{\zeta}=S_{\mathrm{KdV}}(\xi_{\mathrm{KdV}})$ satisfies $\widetilde{\zeta}\in V_{q,R}$ (for q sufficiently small) and, by Lemma 2.7

$$I_q\le\mathcal{E}(\widetilde{\zeta})=q+\mathcal{E}_{\mathrm{KdV}}(\widetilde{\zeta})+\mathcal{O}(q^2).$$

Since $\mathcal{E}_{\mathrm{KdV}}(\widetilde{\zeta}) = q^{\frac{5}{3}}\mathcal{E}_{\mathrm{KdV}}(\xi_{\mathrm{KdV}}) = q^{\frac{5}{3}} I_{\mathrm{KdV}}$, we deduce

$$I_q \leq q + q^{\frac{5}{3}} I_{\mathrm{KdV}} + \mathcal{O}(q^2).$$

We have thus proved (5.16). □

This next result is the first part of Theorem 1.7, which relates the minimizers of $\mathcal{E}$ in $V_{q,R}$ with the minimizers of $\mathcal{E}_{\mathrm{KdV}}$ in U_1.

Theorem 5.5. *Let $q_0 > 0$ be such that Theorem 1.6 and Lemma 5.4 hold. Then for any $q \in (0, q_0)$ and $\zeta \in D_{q,R}$, there exists $x_\zeta \in \mathbb{R}$ such that*

$$\left\| q^{-\frac{2}{3}}\zeta(q^{-\frac{1}{3}}\cdot) - \xi_{\mathrm{KdV}}(\cdot - x_\zeta)\right\|_{H^1} \lesssim q^{\frac{1}{6}} \ ,$$

uniformly with respect to $q \in (0, q_0)$ and $\zeta \in D_{q,R}$.

Proof. Assume that there exists $\epsilon > 0$ and a sequence $\zeta_n \in D_{q_n,R}$ with $q_n \searrow 0$ such that

$$\forall n \in \mathbb{N}, \quad \inf_{x_0\in\mathbb{R}} \left\| q_n^{-\frac{2}{3}}\zeta_n(q_n^{-\frac{1}{3}}\cdot) - \xi_{\mathrm{KdV}}(\cdot - x_0)\right\|_{H^1} \geq \epsilon. \tag{5.17}$$

Denote for simplicity $\xi_n(x) = q_n^{-\frac{2}{3}}\zeta_n(q_n^{-\frac{1}{3}}x)$. From (5.15) in Lemma 5.4, we have

$$I_{q_n} = \mathcal{E}(\zeta_n) = q_n + \mathcal{E}_{\mathrm{KdV}}(\zeta_n) + \mathcal{O}(q_n^2) = q_n + q_n^{\frac{5}{3}}\mathcal{E}_{\mathrm{KdV}}(\xi_n) + \mathcal{O}(q_n^2).$$

By (5.16) in Lemma 5.4, we deduce that

$$\mathcal{E}_{\mathrm{KdV}}(\xi_n) - I_{\mathrm{KdV}} = \mathcal{O}(q_n^{\frac{1}{3}}).$$

In particular $\{\xi_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}_{\mathrm{KdV}}$ satisfying the constraint $(\gamma+\delta)\left\|\xi_n\right\|_{L^2}^2 = 1$. It follows from [2] that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\left\|\xi_n(\cdot - x_n) - \xi_{\mathrm{KdV}}\right\|_{H^1} \to 0,$$

which contradicts (5.17).

The quantitative estimate follows from the argument in [6]. From the above, we may apply Lemma 4.1 therein and define uniquely x_ζ such that $\langle \xi, \xi_{\mathrm{KdV}}(\cdot - x_\zeta)\rangle = 0$, where we denote $\xi = q^{-\frac{2}{3}}\zeta(q^{-\frac{1}{3}}\cdot)$. Following the above estimates and [6, Lemma 5.2], we find

$$\left\|\xi - \xi_{\mathrm{KdV}}(\cdot - x_\zeta)\right\|_{H^1}^2 \lesssim \mathcal{E}_{\mathrm{KdV}}(\xi) - \mathcal{E}_{\mathrm{KdV}}(\xi_{\mathrm{KdV}}) \lesssim q^{\frac{1}{3}}.$$

This concludes the proof. □

Next we prove the second part of Theorem 1.7, which relates the Lagrange multipliers α with the one of the KdV equation, α_0.

Theorem 5.6. *The number α, defined in Theorem 1.6, satisfies*

$$\alpha + 1 = q^{\frac{2}{3}}\alpha_0 + \mathcal{O}(q^{\frac{5}{6}}),$$

uniformly over $D_{q,R}$.

Proof. By Lemmas 2.7 and 5.3, we have

$$\langle \mathrm{d}\mathcal{E}(\zeta), \zeta\rangle = 2q + q^{\frac{5}{3}}\langle \mathrm{d}\mathcal{E}_{\mathrm{KdV}}(S_{\mathrm{KdV}}^{-1}(\zeta)), S_{\mathrm{KdV}}^{-1}(\zeta)\rangle + \mathcal{O}(q^{\frac{7}{3}}).$$

By Theorem 5.5 there exists x_ζ such that $\left\|S_{\mathrm{KdV}}^{-1}(\zeta) - \xi_{\mathrm{KdV}}(\cdot - x_\zeta)\right\|_{H^1} = \mathcal{O}(q^{\frac{1}{6}})$ as $q \searrow 0$. This implies that

$$\langle \mathrm{d}\mathcal{E}_{\mathrm{KdV}}(S_{\mathrm{KdV}}^{-1}(\zeta)), S_{\mathrm{KdV}}^{-1}\zeta\rangle - \langle \mathrm{d}\mathcal{E}_{\mathrm{KdV}}(\xi_{\mathrm{KdV}}), \xi_{\mathrm{KdV}}\rangle = \mathcal{O}(q^{\frac{1}{6}}) \text{ as } q \searrow 0$$

and therefore

$$\langle \mathrm{d}\mathcal{E}(\zeta), \zeta\rangle = 2q + q^{\frac{5}{3}}\langle \mathrm{d}\mathcal{E}_{\mathrm{KdV}}(\xi_{\mathrm{KdV}}), \xi_{\mathrm{KdV}}\rangle + \mathcal{O}(q^{\frac{11}{6}}).$$

Now recall the Euler–Lagrange equations (1.10) and (5.4), which yield immediately

$$2\alpha(\zeta)\, q = -\langle \mathrm{d}\mathcal{E}(\zeta), \zeta\rangle,$$
$$2\alpha_0 = -\langle \mathrm{d}\mathcal{E}_{\mathrm{KdV}}(\xi_{\mathrm{KdV}}), \xi_{\mathrm{KdV}}\rangle,$$

and the result follows. □

6. Numerical Study

In this section, we provide numerical illustrations of our results as well as some numerical experiments for situations which are not covered by our results. We first describe our numerical scheme, before discussing the outcome of these simulations.

6.1. Description of the Numerical Scheme

Our numerical scheme computes solutions for (1.8) for given value of c (and hence does not not follow the minimization strategy developed in this work). Because we seek smooth localized solutions and our operators involve Fourier multipliers, it is very natural to discretize the problem through spectral methods [44]. We are thus left with the problem of finding a root for a nonlinear function defined in a finite (but large) dimensional space. To this aim, we employ the Matlab script `fsolve` which implements the so-called trust-region dogleg algorithm [19] based on Newton's method. For an efficient and successful outcome of the method, it is important to have a fairly precise initial guess. To this aim, we use the exact solution of the Green–Naghdi model, which is either explicit (in the one-layer situation [42]) or obtained as the solution of an ordinary differential equation (in the bi-layer situation [17,38]) that we solve numerically. Our solutions are compared with the corresponding ones of the full Euler system. To compute the latter, we use the Matlab script developed by Per-Olav Rusås and documented in [26] in the bilayer configuration while in the one-layer case, the Matlab script of Clamond and Dutykh [18] offer faster and more accurate results.

6.2. Two-Layer Setting

The solitary-wave solutions of the Miyata–Choi–Camassa system have been studied in the original papers of [17,38]. In particular we know that for a given amplitude, or a given velocity, there exists at most one solitary wave (up to spatial translations). The solitary waves are of elevation if $\delta^2 - \gamma > 0$, of depression if $\delta^2 - \gamma < 0$, and do not exist if $\delta^2 = \gamma$. Contrarily to the one-layer situation, the bilayer Green–Naghdi model admits solitary waves only for a finite range of velocities (resp. amplitudes), $c \in (1, c_{\max}(\gamma, \delta))$ (resp. $|a| \in (0, a_{\max}(\gamma, \delta))$). With our choice of parameters (namely $\gamma = 1, \delta = 1/2$), one has

$$c_{\max} = \sqrt{1 + 1/8} \approx 1.06066 \quad \text{and} \quad |a_{\max}| = 1/2.$$

As the velocity approaches $c_{\max}$, the solitary waves broadens and its mass keeps increasing. These type of profiles or often referred to as "table-top" profiles, and lead to bore profiles in the limit $c \to c_{\max}$.

When the velocity is small the numerically computed solitary wave solutions of the bilayer original ($\mathsf{F}_i = 1$) and full dispersion ($\mathsf{F}_i = \mathsf{F}_i^{\mathrm{imp}}$) Green–Naghdi systems and the one of the water waves systems (and to a lesser extent the KdV model) agree, so that the curves corresponding to the three former models are indistinguishable in see Fig. 2a. For larger velocities, as in Fig. 2b, the numerically computed solitary wave solutions of the Green–Naghdi and water waves systems is very different from the sech^2 profile of the solitary wave solution to the Korteweg–de Vries equation. It is interesting to see that both the original and full dispersion Green–Naghdi models offer good approximations, even in this "large velocity" limit (the normalized l^2 difference of the computed solutions is $\approx 2.10^{-3}$ in both cases). This means that the

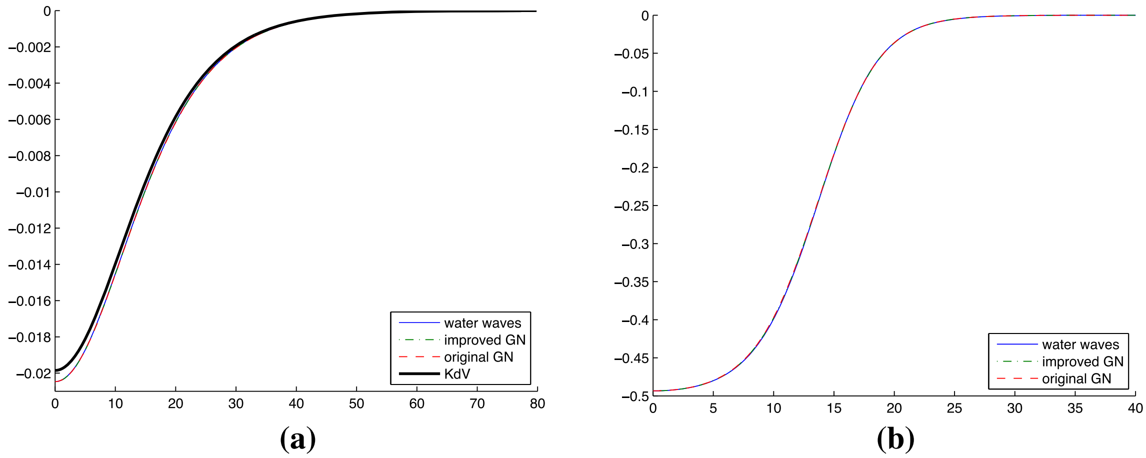

FIG. 2. Comparison of the bilayer Green–Naghdi models and the water waves system ($\gamma = 1, \delta = 1/2$). **a** Small velocity, $c = 1.005$ and **b** large velocity, $c = 1.06065$

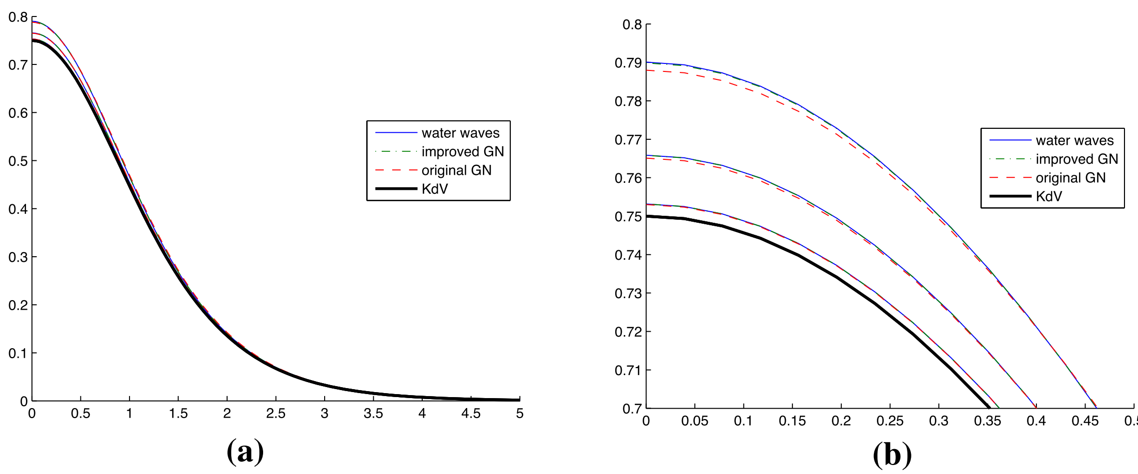

FIG. 3. Comparison of the solutions of the KdV and Green–Naghdi models and the water waves system in the one-layer setting ($\gamma = 0, \delta = 1$). **a** Rescaled solitary waves for $c = 1.025, 1.01, 1.002$ and **b** close-up

internal solitary wave keeps a long-wave feature even for large velocities. These observations were already documented and corroborated by laboratory experiments in [10, 26, 37].

6.3. One-Layer Setting

In the one-layer setting, the script by Clamond and Dutykh [18] allows to have a very precise numerical computation of the water waves solitary solution, from which the numerical solutions of the Green–Naghdi models can be compared. In this setting, namely $\gamma = 0$ and $\delta = 1$, we have an explicit solution for the Green–Naghdi model [42]:

$$\zeta_{\rm GN}(x) = (c^2 - 1)\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{3\frac{c^2-1}{c^2}}x\right) = c^2 \zeta_{\rm KdV}(x).$$

In Fig. 3, we compute the solitary waves for our models with different (small) values of the velocity, rescaled by $S_{\rm KdV}^{-1}$. One clearly sees, as predicted by Theorem 1.7 and the above formula, that the solitary

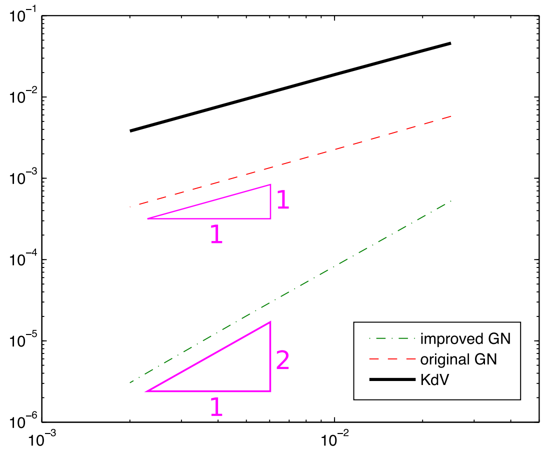

FIG. 4. Convergence rate. Log–log plot of the normalized l^2 norm of the error as a function of $c-1$

waves converge towards $\xi_{\rm KdV}$ after rescaling, as $c \searrow 1$. One also sees that the water waves solution is closer to the one predicted by the model with full dispersion than the original Green–Naghdi model. Figure 4 shows that the convergence rate is indeed quadratic for the full dispersion model whereas it is only linear for the original Green–Naghdi model (and therefore only qualitatively better than the KdV model).

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Compliance with ethical standards
Conflict of interest The authors declare that they have no conflict of interest.

Appendix Paradifferential Calculus

The definitions and properties below are collected from [36]; see also [5,13] for relevant references.

Definition A.1. (*Symbols*) Given $m \in \mathbb{R}$ and $r \geq 0$, we denote Γ^m_r the space of distributions $a(x,\xi)$ on $\mathbb{R}^2$ such that for almost any $x \in \mathbb{R}$, $\xi \mapsto a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R})$, and

$$\forall \alpha \in \mathbb{N}, \exists C_\alpha > 0 \quad \text{such that} \quad \forall \xi \in \mathbb{R}, \quad \big\|\partial_\xi^\alpha a(\cdot,\xi)\big\|_{W^{r,\infty}} \leq C_\alpha (1+|\xi|)^{m-\alpha},$$

where $W^{r,\infty}$ denote the Hölder space (Lipschitz for integer values).

Below, we use an *admissible cut-off* function ψ in the sense of [36, Definition 5.1.4] and define paradifferential operators as follows (the constant factor depends on the choice of convention for the Fourier transform).

Definition A.2. (*Paradifferential operators*) For $a \in \Gamma^m_0$ and $u \in \mathcal{S}(\mathbb{R})$, we define

$$T_a u(x) \stackrel{\rm def}{=} \frac{1}{\sqrt{2\pi}} \left\langle \hat{u}(\cdot), e^{{\rm i}x\cdot} \psi(D,\cdot) a(x,\cdot) \right\rangle_{(\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R}))},$$

where $\psi(D,\xi)$ is the Fourier multiplier associated with $\psi(\eta,\xi)$ (here, ξ is a parameter). The operator is defined for $u \in H^s(\mathbb{R})$ by density and continuous linear extension.

The following lemma is a direct application of the above definitions [36, Theorem 5.1.15].

Lemma A.3. *For any $r \geq 0$ and $a \in \Gamma_r^m \subset \Gamma_0^m$, and for all $s \in \mathbb{R}$, the operator T_a extends in a unique way to a bounded operator from H^{s+m} to H^s.*

If $a(\xi)$ is a symbol independent of x, then $T_a = a(D)$, the corresponding Fourier multiplier.

The main tool we use is the following composition property [36, Theorem 6.1.1].

Lemma A.4. *Let $a \in \Gamma_r^m$ and $b \in \Gamma_r^{m'}$ where $0 < r \leq 1$. Then $ab \in \Gamma_r^{m+m'}$ and $T_aT_b - T_{ab}$ is a bounded operator from $H^{s+m+m'-r}$ to H^s, for any $s \in \mathbb{R}$.*

Of particular interest is the case when the symbol $a(x) \in L^\infty$ is independent of ξ. The admissible cut-off function can be constructed so that the paraproduct $T_a u$ corresponds to a standard Littlewood-Paley decomposition of the product au. This allows to show that $au - T_a u$ is a smoothing operator provided that a is sufficiently regular.

Lemma A.5. *Let $v \in H^s$ and $u \in H^t$, and $r \geq 0$. Then $uv - T_v u \in H^r$ provided that $s + t \geq 0$, $s \geq r$ and $s + t > r + 1/2$.*

The definitions of the paraproduct in [13] and [36] differ slightly but it is not hard to show that [13, Theorem 2.4.1] still holds for the paraproduct as it is defined in [36], and Lemma A.5 follows directly from this theorem.

We conclude with the following lemma, displayed in [36, Theorem 5.2.4]

Lemma A.6. *Let $G \in \mathcal{C}^\infty(\mathbb{R})$ be such that $G(0) = 0$. If $u \in H^s$ with $s > 1/2$, then $G(u) - T_{G'(u)}u \in H^{2s-1/2}$.*

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