Stability and instability of traveling wave solutions to scalar balance laws

V. Duchêne *

June 18, 2020

1 Presentation of the results

In the present contribution we discuss the dynamic stability of traveling wave solutions –including constant equilibria– to first order scalar hyperbolic balance laws

$$\partial_t u + \partial_x\big(f(u)\big) = g(u), \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\,, \tag{1.1}$$

where f and g are regular real functions, accounting respectively for advection and reaction processes. Our discussion is based on the recent works of the author and L.M. Rodrigues, [3, 4].[1] Our aim here is to provide an overview of the results therein in a homogenized framework as well as key ingredients of the proofs, while the reader is referred to the original works for comprehensive results and detailed proofs.

We will prove the large-time asymptotic orbital stability under regular perturbations of piecewise regular entropic traveling wave solutions under non-degeneracy hypotheses and sign criteria at key locations of the wave, namely infinities, discontinuities and characteristic points. We also show the spectral and nonlinear instability of bounded piecewise regular traveling waves satisfying the reverse sign criteria.

An important feature of our results is that we measure stability in strong topologies,[2] obstructing in particular the emergence of additional discontinuities from piecewise regular initial data. That this is conceivable is of course due to the presence of the source term, g, since it is well-known that the solution to scalar conservative laws —that is setting $g \equiv 0$ in (1.1)— emerging from any smooth and decaying at infinity perturbation of a constant state will inevitably lead to the occurrence of a gradient catastrophe in finite time. This is to be compared with the following example:

$$\partial_t u + \partial_x\left(\alpha \frac{u^2}{2}\right) = -\beta\, u$$

with $\alpha \in \mathbb{R}$, $\beta > 0$. Using the method of characteristics, one easily obtain the global-in-time existence of a classical solution as well as exponential decay for any $\mathcal{C}^1(\mathbb{R})$ initial data satisfying $\alpha\,\partial_x(u(0,\cdot)) \geq -\beta$. Hence asymptotic large-time stability of the trivial equilibrium for initial perturbations with sufficiently small derivative holds when $\beta > 0$, while instability is easily seen to hold —as in the purely reactive case, $\alpha = 0$— when $\beta < 0$. We aim at providing a comprehensive theory encompassing general advection and reaction terms as well as general traveling wave solutions.

*Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France (vincent.duchene@univ-rennes1.fr).

[1] It goes without saying that the author of these lines is sole responsible for any flaw in the present document.

[2] This is in fact the key distinction with earlier works on the subject [5, 7–11, 13–16], which describe large-time dynamics in $L^\infty(\mathbb{R})$ topology. Our proofs are also radically different: while the previous references rely on generalized characteristics of Dafermos [2], we employ tools of spectral analysis which are less devoted to the specific case of scalar hyperbolic balance laws. We let the reader refer to [3] for a more detailed discussion.

The outline of the present document is as follows. In Section 1.1, we classify, under non-degeneracy assumptions of the functions f and g, the (strictly) entropic piecewise regular traveling wave solutions to (1.1). We then describe precisely the notions of stability and instability which are used in our results, in Section 1.2. In Section 2 we present three mechanisms of instability, which allow to narrow down the possibly stable waves to a handful of candidates, classified in constants, fronts, shocks and composite waves. In Section 3, we establish the stability of these candidates. Section 4 contains a conclusion and additional comments.

Let us introduce a few notations. For $n \in \mathbb{N}^\star$, $\Omega \subset \mathbb{R}^n$, and $1 \leq p \leq \infty$, $L^p(\Omega)$ is the usual Lebesgue set of p-integrable (or essentially bounded if $p = \infty$) real functions on Ω, and $L^p_{\rm loc}(\Omega)$ is the set of locally p-integrable (or locally essentially bounded if $p = \infty$) real functions on Ω. If $\Omega \subset \mathbb{R}^n$ is connected and $k \in \mathbb{N}$, $W^{k,p}(\Omega)$ (resp. $\mathcal{C}^k(\Omega)$, resp. $BUC^k(\Omega)$) is the set of functions such that derivatives up to the order k belong to $L^p(\Omega)$ (resp. are continuous, resp. are bounded and uniformly continuous). If $\Omega \subset \mathbb{R}^n$ is the disjoint union of connected sets, then $W^{k,p}(\Omega)$ (resp. $\mathcal{C}^k(\Omega)$, resp. $BUC^k(\Omega)$) denotes the set of functions such that the restriction to each connected component, $\check\Omega$, belongs to $W^{k,p}(\check\Omega)$ (resp. $\mathcal{C}^k(\check\Omega)$, resp. $BUC^k(\check\Omega)$). For $D \subset \mathbb{R}$ a closed discrete set, $\ell^\infty(D)$ is the set of bounded real functions on D. We denote

$$\mathcal{C}^{k+}(\mathbb{R}) = \left\{ f \in \mathcal{C}^k(\mathbb{R}) \ : \ r \mapsto \max_{|x-y|\leq r} \tfrac{|g^{(k)}(x)-g^{(k)}(y)|}{r} \in L^1_{\rm loc}(\mathbb{R}) \right\}.$$

These spaces are endowed with natural norms and topological structure associated with their definition.

1.1 Classification of traveling waves

In this section we classify bounded piecewise regular traveling wave solutions to (1.1), under non-degeneracy assumptions on the real functions $f \in \mathcal{C}^2(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$, and entropy conditions introduced thereafter.

Definition 1.1. *A* piecewise regular traveling wave *solution to* (1.1) *is a weak solution in the form*

$$u : (t,x) \mapsto \underline{U}(x - \sigma\, t)$$

with $(\underline{U}, \sigma) \in \mathcal{C}^1(\mathbb{R} \setminus D) \times \mathbb{R}$ *where* D *is a (possibly empty) closed discrete set.*

For $(\underline{U}, \sigma, D)$ defining a piecewise regular traveling wave, $\mathbb{R} \setminus D$ is a union of disjoint open intervals, and

$$\forall x \in \mathbb{R} \setminus D, \quad \big(f'(\underline{U}(x)) - \sigma\big)\underline{U}'(x) = g(\underline{U}(x)). \tag{1.2}$$

Solving this scalar differential equation motivates the following non-degeneracy assumptions.

Definition 1.2. *We say that* $f \in \mathcal{C}^2(\mathbb{R})$ *and* $g \in \mathcal{C}^1(\mathbb{R})$ *are* non-degenerate *if zeroes of* g *are isolated, and*

i. for any $u_\star$ *such that* $g(u_\star) = 0$*, one has* $g'(u_\star) \neq 0$ *and* $f''(u_\star) \neq 0$*;*

ii. for any $u_\star \neq u_\dagger$ *such that* $g(u_\star) = g(u_\dagger) = 0$*, one has* $f'(u_\star) \neq f'(u_\dagger)$*.*

Proposition 1.3. *Let* $f \in \mathcal{C}^2(\mathbb{R})$*,* $g \in \mathcal{C}^1(\mathbb{R})$ *be non-degenerate and* $(\underline{U}, \sigma, D)$ *define a piecewise regular traveling wave solution to* (1.1)*. Then*

- *If* $u_\star \in \underline{U}(\mathbb{R} \setminus D)$ *is a characteristic value, that is* $f'(u_\star) = \sigma$*, then* $g(u_\star) = 0$*;*
- $\underline{U}$ *is either constant or strictly monotonic on connected components of* $\mathbb{R} \setminus D$*;*
- *For any* $d \in D$*,* $\underline{U}$ *possesses left and right limits,* $\underline{U}(d_-) \in \mathbb{R} \cup \{-\infty, +\infty\}$ *and* $\underline{U}(d_+) \in \mathbb{R} \cup \{-\infty, +\infty\}$*;*
- *if a connected component is not lower (resp. upper) bounded then* $\underline{U}$ *has a limit* $u_{-\infty} \in \mathbb{R} \cup \{-\infty, +\infty\}$ *at* $-\infty$ *(resp.* $u_{+\infty} \in \mathbb{R} \cup \{-\infty, +\infty\}$ *at* $+\infty$*). If* $u_{-\infty} \in \mathbb{R}$ *(resp.* $u_{+\infty} \in \mathbb{R}$*), then* $g(u_{-\infty}) = 0$*, (resp.* $g(u_{+\infty}) = 0$*) and either* $\underline{U}$ *is constant on the component, or* $f'(u_{-\infty}) \neq \sigma$ *(resp.* $f'(u_{+\infty}) \neq \sigma$*).*

Proof. The first item is a consequence of (1.2), and one has $f''(u_\star) \neq 0$ and $g'(u_\star) \neq 0$ by Definition 1.2. Hence

$$\forall x \in \mathbb{R} \setminus D, \quad \underline{U}'(x) = 0 \quad \text{or} \quad \underline{U}'(x) = F(\underline{U}(x))$$

where for any J connected component of $\mathbb{R} \setminus D$, $F : \underline{U}(J) \to \mathbb{R}$ is the map defined by

$$\forall u \in \underline{U}(J), \qquad F(u) = \begin{cases} \frac{g(u)}{f'(u)-\sigma} & \text{if } f'(u) - \sigma \neq 0, \\ \frac{g'(u)}{f''(u)} & \text{otherwise.} \end{cases}$$

The remaining items follow from monotonicity and properties of solutions to scalar differential equations. □

A bounded piecewise regular traveling wave defined by $(\underline{U}, \sigma, D)$ possesses left and right limits at any discontinuous point $d \in D$, and hence should satisfy the Rankine-Hugoniot condition, that is

$$\sigma[\underline{U}]_d = [f(\underline{U})]_d\,. \tag{1.3}$$

Above, and thereafter, we denote for $h \in \mathcal{C}^1(V \setminus \{x\})$ where V is a neighborhood of x:

$$h(x_\pm) = \lim_{\delta \searrow 0} h(x \pm \delta) \quad \text{and} \quad [h]_x = h(x_+) - h(x_-)\,.$$

We will assume henceforth that the solution is entropic through (strict) Oleinik's conditions.

Definition 1.4. *We say that a bounded piecewise regular traveling wave solution to* (1.1) *defined by* $(\underline{U}, \sigma, D)$ *is* entropic *if for any* $d \in D$,

$$f'(\underline{U}(d_-)) > \sigma > f'(\underline{U}(d_+)) \tag{1.4}$$

and for any v *strictly between* $\underline{U}(d_+)$ *and* $\underline{U}(d_-)$,

$$\frac{f(v) - f(\underline{U}(d_-))}{v - \underline{U}(d_-)} > \frac{f(v) - f(\underline{U}(d_+))}{v - \underline{U}(d_+)}. \tag{1.5}$$

Remark 1.5. *Let us recall that by the theory due to Kružkov [6], there exists a unique bounded local-in-time entropy-admissible weak solution to* (1.1) *emerging from any bounded initial data. Oleinik's conditions ensure that the piecewise regular traveling wave solution is indeed entropy-admissible, but are also essential to its stability. Indeed, a spectral analysis of the operator* $\mathcal{L}$ *defined below reveals the role of Lax's entropy condition* (1.4) *to the spectral stability of a discontinuous traveling wave, and to the well-posedness of the corresponding linearized dynamics. Condition* (1.5) *would also be crucial to the stability properties of discontinuous traveling waves if one allowed —which is not the case in this work— perturbations breaking the large shock into a "sum" of smaller subshocks; see [1, Remark 4.7] for a more detailed discussion.*

The strict entropy conditions provides useful information on admissible piecewise regular traveling waves.

Proposition 1.6. *Let* $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$ *be non-degenerate and* $(\underline{U}, \sigma, D)$ *define a bounded, piecewise regular entropic traveling wave solution to* (1.1). *Then the following holds.*

- *On any bounded connected component of* $\mathbb{R} \setminus D$, $\underline{U}$ *is strictly monotonic, passes through a characteristic value* $u_\star$ *and* $g'(u_\star) > 0$.
- *On a connected component of* $\mathbb{R} \setminus D$ *bounded from above but not from below,*

  either $\underline{U}$ *is constant with value* $\underline{u} = u_{-\infty}$, *and* $f'(u_{-\infty}) - \sigma > 0$;

  or $\underline{U}$ *is strictly monotonic, possesses a limit* $u_{-\infty} \in \mathbb{R}$ *at* $-\infty$ *with* $g(u_{-\infty}) = 0$, *passes through a characteristic value if and only if* $g'(u_{-\infty}) < 0$, *and* $\operatorname{sgn}(f'(u_{-\infty}) - \sigma) = \operatorname{sgn}(g'(u_{-\infty}))$.

- *On a connected component of* $\mathbb{R} \setminus D$ *bounded from below but not from above,*

  either $\underline{U}$ *is constant with value* $\underline{u} = u_{+\infty}$*, and* $f'(u_{+\infty}) - \sigma < 0$*;*

  or $\underline{U}$ *is strictly monotonic, possesses a limit* $u_{+\infty} \in \mathbb{R}$ *at* $+\infty$ *with* $g(u_{+\infty}) = 0$*, passes through a characteristic value if and only if* $g'(u_{+\infty}) < 0$*, and* $\operatorname{sgn}(f'(u_{+\infty}) - \sigma) = -\operatorname{sgn}(g'(u_{+\infty}))$*.*

- *If* $D = \emptyset$ *and* $\underline{U}$ *is not constant, then* $\underline{U}$ *is strictly monotonic, possesses limits* $u_{\pm\infty} \in \mathbb{R}$ *at* $\pm\infty$ *satisfying* $g(u_{\pm\infty}) = 0$*, passes through a characteristic value if and only if* $g'(u_{-\infty})g'(u_{+\infty}) > 0$ *and* $f'(\underline{U}) - \sigma$ *has the sign of* $\mp g'(u_{\pm\infty})$ *near* $\pm\infty$*.*

Proof. Since we know the sign of $f'(\underline{U}) - \sigma$ near discontinuities of $\underline{U}$ by (1.4), we only need to connect its sign to the sign of $g'(\underline{U})$ near $\pm\infty$ or near a characteristic point, that is $x_\star \in \mathbb{R} \setminus D$ such that $f'(\underline{U}(x_\star)) = \sigma$. At a characteristic point $x_\star$, we have $f''(\underline{U}(x_\star))\, \underline{U}'(x_\star) = g'(\underline{U}(x_\star))$ thus

$$f'(\underline{U}(x)) - \sigma \overset{x \to x_\star}{\sim} g'(\underline{U}(x_\star))\,(x - x_\star)\,.$$

Near $\pm\infty$, if $\underline{U}$ is defined but not constant, the existence of finite limits stems from monotonicity and boundedness, $\underline{U}'$ does not vanish near $\pm\infty$ and

$$f'(\underline{U}(x)) - \sigma \;=\; \frac{g(\underline{U}(x))}{\underline{U}'(x)} \overset{x \to \pm\infty}{\sim} g'(u_{\pm\infty})\,\frac{\underline{U}(x) - u_{\pm\infty}}{\underline{U}'(x)}\,.$$

The claim easily follows. $\square$

Using Proposition 1.6, we may classify bounded piecewise regular entropic traveling wave solution to (1.1).

Corollary 1.7. *Let* $f \in \mathcal{C}^2(\mathbb{R})$*,* $g \in \mathcal{C}^1(\mathbb{R})$ *be non-degenerate and* $(\underline{U}, \sigma, D)$ *define* $u : (t,x) \mapsto \underline{U}(x - \sigma t)$ *a bounded, piecewise regular entropic traveling wave solution to* (1.1)*. Then either*

1. $D = \emptyset$ *and* $\underline{U} \equiv \underline{u} \in \mathbb{R}$ *with* $g(\underline{u}) = 0$*. The value of* $\sigma \in \mathbb{R}$ *is irrelevant. We say that* u *is a constant* equilibrium*.*
2. $D = \emptyset$ *and* $\underline{U}$ *is strictly monotonic, with finite limits* $u_{\pm\infty}$ *at* $\pm\infty$*. One has* $g(u_{-\infty}) = g(u_{+\infty}) = 0$*. We say that* u *is a* front*.*
3. $D = \{d\}$ *and* $\underline{U}$ *is constant on* $(-\infty, d)$ *and on* $(d, +\infty)$*, and takes two different values* $\underline{u}_- \neq \underline{u}_+$*. One has* $g(\underline{u}_-) = g(\underline{u}_+) = 0$ *and* $\sigma = \frac{f(\underline{u}_+) - f(\underline{u}_-)}{\underline{u}_+ - \underline{u}_-}$*. We say that* u *is a* shock*.*
4. *In any other case,* $D \neq \emptyset$ *and* $\underline{U}$ *is strictly monotonic on at least one connected components of* $\mathbb{R} \setminus D$*, and is constant on at most two (unbounded) connected component of* $\mathbb{R} \setminus D$*. We say that* u *is a* composite *traveling wave. This contains in particular periodic traveling waves.*

We may now describe our results. Of course, notions of stability and instability are detailed in precise statements thereafter; they are introduced and discussed in the following section.

Theorem (rough statement)**.** *Let* f *and* g *be sufficiently regular and non-degenerate, and* $(\underline{U}, \sigma, D)$ *define a bounded piecewise regular entropic traveling wave solution to* (1.1)*. Then* $u : (t,x) \mapsto \underline{U}(x - \sigma t)$ *is stable if and only if*

- $\underline{U}$ *has finite limits* $u_{\pm\infty}$ *at* $\pm\infty$ *and* $g'(u_{-\infty}) < 0$*,* $g'(u_{+\infty}) < 0$*;*
- *on connected components such that* $\underline{U}$ *is strictly monotonous,* $\underline{U}$ *passes through a characteristic value* $u_\star \in \mathbb{R}$*, and* $g'(u_\star) > 0$*;*
- *for any* $d \in D$*,* $\frac{[g(\underline{U})]_d}{[\underline{U}]_d} \leq 0$*.*

In other words, and following the terminology of Corollary 1.7, stable piecewise regular entropic traveling wave solutions to (1.1) *consist in*

1. *constant equilibria* $\underline{u} \in \mathbb{R}$ *such that* $g(\underline{u}) = 0$ *and* $g'(\underline{u}) < 0$ *(we say the equilibria are dissipative);*
2. *fronts taking a characteristic value* $u_\star \in \mathbb{R}$*, and such that* $g'(u_\star) > 0$*;*
3. *(strictly) entropic shocks between two dissipative equilibria;*
4. *composite waves satisfying 1. (resp. 2.) on connected components where* $\underline{U}$ *is constant (resp. strictly monotonous), and* $\frac{[g(\underline{U})]_d}{[\underline{U}]_d} < 0$ *and* (1.4)-(1.5) *at any discontinuity, d.* $\underline{U}$ *is constant on one or two (necessarily unbounded) connected components and strictly monotonous on exactly one component.*

Remark 1.8. *That the two assertions of the Theorem are equivalent is a direct consequence of Proposition 1.6. The only non-trivial task consists in ruling out the possibility of* $\underline{U}$ *being strictly monotonous on two consecutive connected components and defining a stable piecewise regular entropic traveling wave solutions. This stems from the fact that if* $\underline{U}$ *is strictly monotonic on two consecutive connected components separated by* $d_0 \in D$*, then it passes through a single characteristic point* $u_\star$ *in both components, by Proposition 1.6 and the non-degeneracy condition. We can then check that the inequalities* $g'(u_\star) > 0$ *and* $\frac{[\,g(\underline{U})\,]_{d_0}}{[\,\underline{U}\,]_{d_0}} \leq 0$ *are incompatible.*

1.2 Notions of stability

Let us now clarify what is meant by "stability" in this work. As aforementioned, closedness will be described with topologies controlling piecewise smoothness. Moreover, while the stability results will encode control on deformations of shape, some re-synchronization of positions is allowed. This is essential in the presence of discontinuities, but will turn out to be useful as well in the presence of characteristic points. We will also distinguish between the spectral stability described through the spectrum of linearized problems, and dynamic nonlinear stability describing the large-time behavior of solutions to (1.1) with close initial data. The main lesson of this work is that, in our framework, the two notions coincide and that the nonlinear stability under regular perturbations may be decided from a handful of sign conditions encoding spectral stability. While such a statement is familiar in the study of dynamic stability, one should mention that this result is far from obvious for convection/reaction equations, due to the fact that the operators at stake do not offer any regularization effects. As a consequence, it is not sufficient to consider the linearized dynamics about the traveling wave, and our proofs rely instead on decay estimates for all nearby linear dynamics.

With the above discussion in mind, for $(\underline{U}, \sigma, D)$ defining a bounded piecewise regular traveling wave solution to (1.1), we look for entropic solutions u to (1.1) in the form

$$u(t, x + \sigma t + \psi(t, x)) = \underline{U}(x) + \widetilde{u}(t, x) \tag{1.6}$$

with $(\widetilde{u}, \psi)$ small provided they are sufficiently small initially. We always assume that $\psi(t, \cdot)$ is *admissible*, that is $x \in \mathbb{R} \mapsto x + \psi(t, x)$ is increasing and bijective.

Definition 1.9 (Nonlinear stability)**.** *Given functional spaces* $\mathcal{X}, \mathcal{Y}$ *of locally integrable real functions, a bounded piecewise regular traveling wave solution to* (1.1) *defined by* $(\underline{U}, \sigma, d)$ *is* nonlinearly stable *in* $\mathcal{X} \times \mathcal{Y}$ *if for any* $\mathcal{U} \times \mathcal{V}$ *neighborhood of* $\underline{U} \times \{0\}$ *for the* $\mathcal{X} \times \mathcal{Y}$ *topology, there exists* $\mathcal{U}_0 \times \mathcal{V}_0$ *another neighborhood of* $\underline{U} \times \{0\}$ *for the* $\mathcal{X} \times \mathcal{Y}$ *topology such that for any* $(u_0, \psi_0) \in \mathcal{U}_0 \times \mathcal{V}_0$ *such that* ψ_0 *is admissible, the unique global-in-time entropy solution emerging from the initial data defined by* $u(0, x + \psi_0(x)) = u_0(x)$ *satisfies that for any* $t \in \mathbb{R}^+$*, there exists* $\psi(t, \cdot) \in \mathcal{V}$ *admissible such that* $u(t, \cdot + \sigma t + \psi(t, \cdot)) \in \mathcal{U}$*. The traveling wave is* nonlinearly unstable *in* $\mathcal{X} \times \mathcal{Y}$ *otherwise. A traveling wave nonlinearly stable in* $\mathcal{X} \times \mathcal{Y}$ *is* orbitally stable with asymptotic phase *if, additionally, there exists* $\psi_\infty \in \mathcal{Y}$ *admissible such that* $\big(u(t, \cdot + \sigma t + \psi(t, \cdot)), \psi(t, \cdot)\big) \to \big(\underline{U}, \psi_\infty\big)$ *in* $\mathcal{X} \times \mathcal{Y}$ *as* $t \to \infty$*.*

Note that we aim at a space shift $\psi(t,\cdot)$ regular on $\mathbb{R}$ and a shape deformation $\widetilde{u}(t,\cdot)$ regular on $\mathbb{R}\setminus D$ with limits from both sides at each $d\in D$. This leads us to the following restriction on entropy solutions.

Definition 1.10. *We say that $u\in L^1_{\mathrm{loc}}(I\times\mathbb{R})$ is a piecewise regular entropy solution to* (1.1) *on the time interval $I\subset\mathbb{R}$ if there exist a closed discrete set D and $\psi\in\mathcal{C}^1(I\times\mathbb{R})$ admissible for any $t\in I$ such that $(t,x)\mapsto u(t,x+\sigma t+\psi(t,x))\in\mathcal{C}^1(I\times(\mathbb{R}\setminus D))$, and u satisfies* (1.1) *in its regular domain, as well as the Rankine-Hugoniot condition*

$$\big(u_l(t,d)-u_r(t,d)\big)\big(\sigma+\partial_t\psi(t,d)\big)\;=\;f(u_l(t,d))-f(u_r(t,d))$$

and the strict Oleinik entropy conditions

$$\begin{cases}\sigma+\partial_t\psi(t,d)\;>\;f'(u_r(t,d))\,,\\ \frac{f(\tau\,u_l(t,d)+(1-\tau)\,u_r(t,d))-f(u_l(t,d))}{\tau\,u_l(t,d)+(1-\tau)\,u_r(t,d)-u_l(t,d)}>\frac{f(\tau\,u_l(t,d)+(1-\tau)\,u_r(t,d))-f(u_r(t,d))}{\tau\,u_l(t,d)+(1-\tau)\,u_r(t,d)-u_r(t,d)} & \textit{for any}\;\;\tau\in(0,1)\,,\\ f'(u_l(t,d))\;>\;\sigma+\partial_t\psi(t,d)\,.\end{cases}\tag{1.7}$$

where $u_l(t,d)=\lim_{\delta\searrow 0}u(t,d+\sigma t+\psi(t,d)-\delta)$ and $u_r(t)=\lim_{\delta\searrow 0}u(t,d+\sigma t+\psi(t,d)+\delta)$.

Notice that we assume in Definition 1.10 that discontinuities do not vanish or emerge, and discontinuity paths do not touch on the time interval I. Using identity (1.6) and chain rules, one can check that u being a piecewise regular entropy solution reduces to interior equations

$$\begin{aligned}\partial_t(\widetilde{u}-\psi\underline{U}')+\partial_x\big((f'(\underline{U})-\sigma)(\widetilde{u}-\psi\underline{U}')\big)-g'(\underline{U})(\widetilde{u}-\psi\underline{U}')\\ =-\partial_x(f(\underline{U}+\widetilde{u})-f(\underline{U})-f'(\underline{U})\widetilde{u})+g(\underline{U}+\widetilde{u})-g(\underline{U})-g'(\underline{U})\widetilde{u}\\ +\partial_x\psi\,(g(\underline{U}+\widetilde{u})-g(\underline{U}))-\partial_t(\partial_x\psi\,\widetilde{u})+\partial_x(\partial_t\psi\,\widetilde{u})\end{aligned}$$

on $\mathbb{R}\setminus D$, and at any $d\in D$ the Rankine-Hugoniot condition

$$\partial_t\psi\,[\,\underline{U}\,]_d\;-\;[\,(f'(\underline{U})-\sigma)\widetilde{u}\,]_d\;=\;[\,f(\underline{U}+\widetilde{u})-f(\underline{U})-f'(\underline{U})\widetilde{u})\,]_d\;-\;\partial_t\psi\,[\,\widetilde{u}\,]_d$$

and the Oleinik entropy conditions which we omit to write down. Indeed, since we only consider waves satisfying strict entropy condition, they do not show up at the linearized level. The above suggests to consider the spectral problem associated with the linearized equations, that is

$$\begin{aligned}&\lambda\,(\check{u}-\check{\psi}\underline{U}')+\partial_x((f'(\underline{U})-\sigma)(\check{u}-\check{\psi}\underline{U}'))-g'(\underline{U})(\check{u}-\check{\psi}\underline{U}')=A+\partial_x(B) &&\text{on } \mathbb{R}\setminus D\,,\\ &\lambda\check{\psi}\,[\,\underline{U}\,]_d\;-\;[\,(f'(\underline{U})-\sigma)(\check{u}-\check{\psi}\underline{U}')\,]_d\;-\;\check{\psi}\,[\,(f'(\underline{U})-\sigma)\underline{U}'\,]_d=[\,-B\,]_d &&\text{at any } d\in D\,,\end{aligned}$$

with $(\check{\psi},\check{u})$ playing the role of the value at λ of the Laplace transform in time of $(\psi,\widetilde{u})$.

For the sake of tractability we relax the foregoing problem into the problem of the determination of the spectrum of a given operator. To do so we choose $\mathcal{X}$ a functional space of locally integrable functions on $\mathbb{R}\setminus D$ and $\mathcal{Y}$ a space of functions on D. We assume that any $w\in\mathcal{X}$ such that $\partial_x((f'(\underline{U})-\sigma)w)-g'(\underline{U})w\in\mathcal{X}$ possesses limits from the left and from the right at any point $d\in D$. Then one may define on $\mathcal{X}\times\mathcal{Y}$, the operator with maximal domain

$$\mathcal{L}(w,(y_d)_{d\in D}):=\left(-\partial_x((f'(\underline{U})-\sigma)w)+g'(\underline{U})w,\left(y_d\frac{[\,(f'(\underline{U})-\sigma)\underline{U}'\,]_d}{[\,\underline{U}\,]_d}\;+\;\frac{[\,(f'(\underline{U})-\sigma)w\,]_d}{[\,\underline{U}\,]_d}\right)_{d\in D}\right)\,.$$

Definition 1.11 (Spectral stability)**.** *We call* $\mathcal{X}\times\mathcal{Y}$-spectrum *of the linearization about a piecewise regular traveling wave defined by $(\underline{U},\sigma,D)$ the spectrum of $\mathcal{L}$. We say that the wave defined by $(\underline{U},\sigma,D)$ is* spectrally stable *in $\mathcal{X}\times\mathcal{Y}$ if the $\mathcal{X}\times\mathcal{Y}$-spectrum is contained in the set of complex values with negative real parts, and $\{0\}$.*[3] *We say that the wave defined by $(\underline{U},\sigma,D)$ is* spectrally unstable *if there exists an element of the $\mathcal{X}\times\mathcal{Y}$-spectrum with positive real part.*

[3]Unless $\underline{U}$ is constant and $D=\emptyset$, $(\underline{U}',(1)_{d\in D})\in\mathcal{X}\times\mathcal{Y}$ is an eigenfunction associated with eigenvalue 0, pertaining to the translation invariance of the problem.

2 Instability results

In this section, we exhibit instability mechanisms for piecewise regular traveling wave solutions to (1.1), both at the spectral and nonlinear level.

2.1 Instabilities at infinity

Proposition 2.1. *Let $k \in \mathbb{N}$, $f \in \mathcal{C}^{k+2}(\mathbb{R})$, $g \in \mathcal{C}^2(\mathbb{R}) \cap \mathcal{C}^{k+1}(\mathbb{R})$ be non-degenerate and $(\underline{U}, \sigma, D)$ define a piecewise regular traveling wave solution to* (1.1)*, u. If D is unbounded from above (resp. from below) and $\underline{U}$ admits a limit $u_{+\infty} \in \mathbb{R}$ at $+\infty$ (resp. $u_{-\infty} \in \mathbb{R}$ at $-\infty$), then $g'(u_{+\infty}) + i(f'(u_{+\infty}) - \sigma)\mathbb{R}$ (resp. $g'(u_{-\infty}) + i(f'(u_{-\infty}) - \sigma)\mathbb{R}$) is included in the $\mathcal{X} \times \mathcal{Y}$-spectrum of the linearization about u if for some $I \subset \mathbb{R}$ neighborhood of $+\infty$ (resp. $-\infty$) the norm of $\mathcal{X}$ restricted to smooth functions compactly supported in I is controlled by the $W^{k,p}(I)$-norm and controls the $L^q(I)$-norm, for some $1 \leq p,\ q \leq \infty$ such that $(p, q) \neq (1, \infty)$.*

In particular, if $g'(u_{+\infty}) > 0$ (resp. $g'(u_{-\infty}) > 0$) then u is spectrally unstable in $BUC^k(\mathbb{R} \setminus D) \times \ell^\infty(D)$.

Proof. Since the difference is purely notational we only treat the case where the limit is at $+\infty$. We pick $\chi : \mathbb{R} \to \mathbb{R}$ non zero, smooth and compactly supported. For $\xi \in \mathbb{R}$ and $\varepsilon > 0$, we let

$$w^{(\varepsilon)} \,:\, \mathbb{R} \setminus D \to \mathbb{C},\, x \mapsto e^{-i\,\xi x}\,\chi\left(\varepsilon\, x - \frac{1}{\varepsilon}\right).$$

For ε sufficiently small, $w^{(\varepsilon)}$ is supported in I and

$$\begin{aligned} \|(g'(u_{+\infty}) + i(f'(u_{+\infty}) - \sigma)\xi)w^{(\varepsilon)} + \partial_x((f'(\underline{U}) - \sigma)w^{(\varepsilon)}) - g'(\underline{U})w^{(\varepsilon)}\|_{W^{k,p}(I)} &\lesssim \varepsilon^{1-\frac{1}{p}} \\ \|w^{(\varepsilon)}\|_{L^q(I)} &\gtrsim \varepsilon^{-\frac{1}{q}}, \end{aligned}$$

where we used that $\underline{U} - u_{+\infty}$ and derivatives up to the order $k + 2$ converge exponentially fast to zero at infinity. Hence $(w^{(\varepsilon_n)}, (0)_{d \in D})_{n \in \mathbb{N}}$ with $(\varepsilon_n)_{n \in \mathbb{N}}$ positive and converging to zero defines a Weyl sequence, and the proof is complete. ☐

Proposition 2.2. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$ be non-degenerate and $(\underline{U}, \sigma, D)$ define a bounded piecewise regular entropy-admissible traveling wave solution to* (1.1)*, u. If D is unbounded from above (resp. from below) and $\underline{U}$ admits a limit $u_{+\infty}$ at $+\infty$ (resp. $u_{-\infty}$ at $-\infty$) and $g'(u_{+\infty}) > 0$ (resp. $g'(u_{-\infty}) > 0$), then the following holds. There exists $\delta > 0$, and a sequence $(u_n)_{n \in \mathbb{N}}$ of piecewise regular weak solutions to* (1.1) *defined for $t \in [0, T_n]$ such that for any $I \subset \mathbb{R}$ neighborhood of $+\infty$ (resp. $-\infty$), one has for n sufficiently large*

i. $u_n(0, \cdot) - \underline{U}$ is smooth and compactly supported in I, and for any $k \in \mathbb{N}$ and $1 \leq q \leq \infty$,

$$\left\|u_n(0, \cdot) - \underline{U}\right\|_{W^{k,q}(I)} \to 0 \quad \text{as} \quad n \to \infty,$$

ii. $(t, x) \mapsto u_n(t, \cdot + \sigma t) - \underline{U}(t, \cdot) \in \mathcal{C}^1([0, T_n] \times \mathbb{R})$, has compact support in $[0, T_n] \times I$, and for any admissible $\psi \in \mathcal{C}^0(\mathbb{R})$ and any $1 \leq p \leq \infty$,

$$\left\|u_n(T_n, \cdot + \psi(\cdot)) - \underline{U}\right\|_{L^p(\mathbb{R})} \geq \delta.$$

In particular, u is nonlinearly unstable in $BUC^k(\mathbb{R} \setminus D) \times BUC^{k'}(\mathbb{R})$ for any $(k, k') \in \mathbb{N}^2$.

Proof. Since the difference is purely notational we only treat the case where the limit is at $+\infty$. We set $u_n(0, \cdot) = \underline{U} + w_{\varepsilon_n}$ where $w_\varepsilon = \varepsilon^2\chi(\varepsilon x - \frac{1}{\varepsilon})$ with $\chi : \mathbb{R} \to \mathbb{R}$ non zero, smooth and compactly supported, and $(\varepsilon_n)_{n \in \mathbb{N}}$ is positive and converging to zero. We may construct locally in time the solution u_n via characteristics. For $\varepsilon > 0$ and $x \in (x_\varepsilon, +\infty)$ where $x_\varepsilon := \inf(\mathrm{supp}(w_\varepsilon))$, we define $v_\varepsilon(\cdot, x)$ and $X_\varepsilon(\cdot, x)$ by the initial data $v_\varepsilon(0, x) = (\underline{U} + w_\varepsilon)(x)$ and $X_\varepsilon(0, x) = x$, and the differential equations

$$\partial_t v_\varepsilon(t, x) = g\big(v_\varepsilon(t, x)\big) \quad \text{ and } \quad \partial_t X_\varepsilon(t, x) = f'\big(v_\varepsilon(t, x)\big).$$

For any $\alpha > 0$, there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and for any $t > 0$ such that $r(t) = \sup(\{|v_\varepsilon(s, x) - u_{+\infty}| \; : \; s \in [0, t], x \in (x_\varepsilon, +\infty)\}) \leq \delta_0$, one has

$$|v_\varepsilon - u_{+\infty}|(t, x) \geq |v_\varepsilon(0, x) - u_{+\infty}| e^{(g'(u_{+\infty})-\alpha)t} \quad \text{ and } \quad |\partial_x v_\varepsilon(t, x)| \leq |\partial_x v_\varepsilon(0, x)| e^{(g'(u_{+\infty})+\alpha)t}.$$

Choosing α sufficiently small and lowering ε_0 further if necessary, we deduce that there exists $T_\varepsilon \in (0, +\infty)$ such that $r(T_\varepsilon) = \delta_0$, and $|\partial_x v_\varepsilon| \leq 1$ and $\partial_x X_\varepsilon \geq 1/2$ on $[0, T_\varepsilon] \times (x_\varepsilon, +\infty)$. We then uniquely define u_ε through the relation $u_\varepsilon(t, X_\varepsilon(t, x)) = v_\varepsilon(t, x)$ on $\{(s, y), 0 \leq s \leq T_\varepsilon,\ X_\varepsilon(t, x_\varepsilon) < y < \infty\}$, and set $u_\varepsilon(t, x) = \underline{U}(x - \sigma t)$ for $x \in \mathbb{R} \setminus (X_\varepsilon(t, x_\varepsilon), +\infty)$. We easily verify that $(u_{\varepsilon_n})_{n\in\mathbb{N}}$, for $(\varepsilon_n)_{n\in\mathbb{N}}$ positive, sufficiently small and converging towards zero satisfies the desired properties. $\square$

2.2 Instabilities at characteristic points

Proposition 2.3. *Let $k \in \mathbb{N}^\star$ and $f \in \mathcal{C}^{k+2}(\mathbb{R})$, $g \in \mathcal{C}^{k+1}(\mathbb{R})$ and $(\underline{U}, \sigma, D)$ define a piecewise regular traveling wave solution to* (1.1)*, u. If $x_\star \in \mathbb{R} \setminus D$ is a characteristic point, that is $\underline{U}(x_\star) = u_\star$ with $f'(u_\star) = \sigma$ and $\underline{U}'(x_\star) \neq 0$, then for any $j \in \{1, \ldots, k\}$, $-g'(u_\star)\, j$ belongs to the $\mathcal{X} \times y$-spectrum of the linearization about u provided that $\delta_{x_\star}, \cdots, \delta_{x_\star}^{(j)}$ act continuously on $\mathcal{X}$.*

In particular, if $g'(u_\star) < 0$, then u is spectrally unstable in $BUC^k(\mathbb{R} \setminus D) \times \ell^\infty(D)$.

Proof. It is sufficient to prove that there exists $\widetilde{w}$ a non trivial combination of $\delta_{x_\star}, \cdots, \delta_{x_\star}^{(j)}$ such that

$$-g'(u_\star) j\, \widetilde{w} - (f'(\underline{U}) - \sigma)\partial_x \widetilde{w} - g'(\underline{U})\widetilde{w} \;=\; 0\,.$$

This follows from the fact that $f'(\underline{U}(x)) - \sigma = (x - x_\star)g'(u_\star) + o(x - x_\star)(x \to x_\star)$ and hence for any $j \in \mathbb{N}^\star$

$$-(f'(\underline{U}) - \sigma)\delta_{x_\star}^{(j+1)} - g'(\underline{U})\delta_{x_\star}^{(j)} \;\in\; j\, g'(u_\star)\, \delta_{x_\star}^{(j)} \;+\; \mathrm{span}(\{\delta_{x_\star}, \cdots, \delta_{x_\star}^{(j-1)}\})\,.$$

$\square$

Proposition 2.4. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$ and $(\underline{U}, \sigma, D)$ define a bounded piecewise regular entropy-admissible traveling wave solution to* (1.1)*, u. If $x_\star \in \mathbb{R} \setminus D$ is a characteristic point, that is $\underline{U}(x_\star) = u_\star$ with $f'(u_\star) = \sigma$ and $\underline{U}'(x_\star) \neq 0$, and if $g'(u_\star) < 0$, then the following holds. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ of piecewise regular solutions to* (1.1) *defined for $t \in [0, T_n)$ such that for any $I \subset \mathbb{R}$ neighborhood of $x_\star$, one has for n sufficiently large*

i. $u_n(0, \cdot) - \underline{U}$ is smooth and compactly supported in I, and for any $k \in \mathbb{N}$ and $1 \leq q \leq \infty$,

$$\big\|u_n(0, \cdot) - \underline{U}\big\|_{W^{k,q}(I)} \to 0 \quad \text{ as } \quad n \to \infty,$$

ii. $(t, x) \mapsto u_n(t, \cdot + \sigma t) - \underline{U}(t, \cdot) \in \mathcal{C}^1([0, T_n) \times \mathbb{R}) \cap L^\infty([0, T_n) \times \mathbb{R})$, has support in $[0, T_n) \times I$, and

$$\big\|u_n(t, \cdot + \sigma t)\big\|_{W^{1,\infty}(I)} \to \infty \quad \text{ as } t \to T_n.$$

In particular, u is nonlinearly unstable in $BUC^k(\mathbb{R} \setminus D) \times BUC^{k'}(\mathbb{R})$ for any $(k, k') \in (\mathbb{N}^\star)^2$.

Proof. We pick $\chi : \mathbb{R} \to \mathbb{R}$ a real smooth compactly supported function such that $\chi(x_\star) = 0$ and $\chi'(x_\star) > 0$. For $\varepsilon \in (0,1)$, we denote $\delta_\varepsilon := \frac{-1}{\ln \varepsilon}$, $\chi_\varepsilon := \chi(x_\star + \frac{\cdot - x_\star}{\delta_\varepsilon})$, $x_\varepsilon^- := \inf(\mathrm{supp}(\chi_\varepsilon))$, $x_\varepsilon^+ := \sup(\mathrm{supp}(\chi_\varepsilon))$, and for any $x \in (x_\varepsilon^-, x_\varepsilon^+)$ we define $v_\varepsilon(\cdot, x)$ and $X_\varepsilon(\cdot, x)$ by the initial data $v_\varepsilon(0,x) = \underline{U}(x) + \varepsilon \underline{U}'(x_\star)\chi_\varepsilon(x)$ and $X_\varepsilon(0,x) = x$, and the differential equations

$$\partial_t v_\varepsilon(t,x) = g\big(v_\varepsilon(t,x)\big), \quad \partial_t X_\varepsilon(t,x) = f'\big(v_\varepsilon(t,x)\big).$$

There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in (x_\varepsilon^-, x_\varepsilon^+)$, $\mathrm{sgn}(g(v_\varepsilon(0,x))) = -\,\mathrm{sgn}(v_\varepsilon(0,x) - u_\star)$, and hence $v_\varepsilon(\cdot, x), X_\varepsilon(\cdot, x) \in \mathcal{C}^2(\mathbb{R}^+)$ and $\sup(\{|v_\varepsilon(t,x) - u_\star| \ : \ t \geq 0\}) = |v_\varepsilon(0,x) - u_\star|$. Moreover, $v_\varepsilon(t, x_\star) = \underline{U}(x_\star) = u_\star$ and $X_\varepsilon(t, x_\star) = x_\star - \sigma t$, and hence the identities $\partial_x v_\varepsilon(t,x_\star) = \partial_x v_\varepsilon(0,x_\star) e^{g'(u_\star)t}$ and $\partial_x X_\varepsilon(t,x_\star) = 1 + \frac{f''(u_\star)}{g'(u_\star)} \partial_x v_\varepsilon(0,x_\star)\big(e^{g'(u_\star)t} - 1\big)$ hold for any $t \geq 0$. Since $\partial_x v_\varepsilon(0,x_\star) = \underline{U}'(x_\star)(1 + \frac{\varepsilon}{\delta_\varepsilon}\chi'(x_\star))$, $\underline{U}'(x_\star) = \frac{g'(u_\star)}{f''(u_\star)}$ and $\chi'(x_\star) > 0$ and $g'(u_\star) < 0$, we infer that

$$T_\varepsilon := \sup\big(\big\{t > 0 \ : \ \inf_{s \in [0,t], x \in (x_\varepsilon^-, x_\varepsilon^+)} (\partial_x X_\varepsilon(s,x)) > 0\big\}\big) < \infty.$$

We then uniquely define u_ε by $u_\varepsilon(t, X_\varepsilon(t,x)) = v_\varepsilon(t,x)$ on $\{(s,y), 0 \leq s < T_\varepsilon, X_\varepsilon(t, x_\varepsilon^-) \leq y \leq X_\varepsilon(t, x_\varepsilon^+)\}$, and set $u_\varepsilon(t,x) = \underline{U}(x - \sigma t)$ for $x \in \mathbb{R} \setminus (X_\varepsilon(t, x_\varepsilon^-), X_\varepsilon(t, x_\varepsilon^+))$. We easily verify that $(u_{\varepsilon_n})_{n\in\mathbb{N}}$, for $(\varepsilon_n)_{n\in\mathbb{N}}$ positive, sufficiently small and converging towards zero satisfies the desired properties. $\square$

2.3 Instabilities of shock positions

Proposition 2.5. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$ and $(\underline{U}, \sigma, D)$ define a bounded piecewise regular traveling wave solution to* (1.1)*, u. For any $d_0 \in D$, $\frac{[g(\underline{U})]_{d_0}}{[\underline{U}]_{d_0}}$ belongs to the $\mathcal{X} \times y$-spectrum of the linearization about u. In particular, if $\frac{[g(\underline{U})]_{d_0}}{[\underline{U}]_{d_0}} > 0$ for some $d_0 \in D$, u is spectrally unstable in $BUC^k(\mathbb{R} \setminus D) \times \ell^\infty(D)$ for any $k \in \mathbb{N}$.*

Proof. One readily checks that $(w, (y_d)_{d\in D}) = (0, (\delta_{d,d_0})_{d\in D})$ provides an eigenvector for $\frac{[g(\underline{U})]_{d_0}}{[\underline{U}]_{d_0}}$. $\square$

Proposition 2.6. *Let $k \in \mathbb{N}$, $f \in \mathcal{C}^{k+1}(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^k(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and $(\underline{U}, \sigma, D)$ define a bounded piecewise regular entropy-admissible traveling wave solution to* (1.1)*, u, such that for some $d_0 \in D$, $\frac{[g(\underline{U})]_{d_0}}{[\underline{U}]_{d_0}} > 0$. Then the following holds. For any $I \subset \mathbb{R}$ neighborhood of d_0, there exists $\delta > 0$, a sequence $(u_n)_{n\in\mathbb{N}}$ of piecewise regular solutions to* (1.1) *defined for $t \in [0, T_n]$ and a sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth admissible functions compactly supported on $[0, T_n] \times I$ such that $(t,x) \mapsto u_n(t, x + \sigma t + \psi_n(t,x)) - \underline{U}(x) \in \mathcal{C}^1([0,T_n] \times (\mathbb{R} \setminus D))$ has support in $[0, T_n] \times (I \setminus \{d_0\})$ and*

i. for any $1 \leq q \leq \infty$

$$\big\|\psi_n(0,\cdot)\big\|_{W^{k,q}(I)} \to 0 \quad \textit{and} \quad \big\|u_n(0, \cdot + \psi_n(0,\cdot)) - \underline{U}\big\|_{W^{k,q}(I \setminus \{d_0\})} \to 0 \quad \textit{as } n \to \infty,$$

ii. for any admissible $\psi \in \mathcal{C}^0(\mathbb{R})$ and any $1 \leq p \leq \infty$,

$$\big\|u_n(T_n, \cdot + \psi(\cdot)) - \underline{U}\big\|_{L^p(\mathbb{R})} \geq \delta.$$

Moreover, if the strict entropy conditions holds at $d \in d_0$, then it continues to hold at the shock position $x(t)$ such that $x(t) + \sigma t + \psi_n(t, x(t)) = d_0$. In particular, u is nonlinearly unstable in $BUC^k(\mathbb{R} \setminus D) \times BUC^k(\mathbb{R})$.

Proof. There exists $\eta > 0$ and $\underline{U}_-, \underline{U}_+ \in \mathcal{C}^1([d_0 - \eta, d_0 + \eta])$ such that for any $x \in [d_0 - \eta, d_0)$, $\underline{U}_-(x) = \underline{U}(x)$, for any $x \in (d_0, d_0 + \eta]$, $\underline{U}_+(x) = \underline{U}(x)$, and $\big(f'(\underline{U}_\pm) - \sigma\big)\underline{U}_\pm' = g(\underline{U}_\pm)$ on $[d_0 - \eta, d_0 + \eta]$. We notice that $\Big(\frac{f(\underline{U}_+) - f(\underline{U}_-)}{\underline{U}_+ - \underline{U}_-}\Big)'(d_0) = \frac{[g(\underline{U})]_{d_0}}{[\underline{U}]_{d_0}}$. By continuity, lowering $\eta > 0$ if necessary, we may ensure that

$$\alpha := \inf_{x \in (d_0 - \eta, d_0 + \eta)} \left(\frac{f(\underline{U}_+) - f(\underline{U}_-)}{\underline{U}_+ - \underline{U}_-}\right)'(x) > 0,$$

and, if (1.4) and (1.5) hold at $d = d_0$, for any $x \in [d_0 - \eta, d_0 + \eta]$ and v strictly between $\underline{U}_-(x)$ and $\underline{U}_+(x)$,

$$f'(\underline{U}_-(x)) > \sigma > f'(\underline{U}_+(x)) \quad \text{and} \quad \frac{f(v) - f(\underline{U}_-(x))}{v - \underline{U}_-(x)} > \frac{f(v) - f(\underline{U}_+(x))}{v - \underline{U}_+(x)}.$$

For any $\varepsilon \in (-\eta, \eta)$, we define $u_\varepsilon(t, \cdot)$ for $t \in [0, T_\varepsilon]$ as

$$u_\varepsilon(t, x) = \begin{cases} \underline{U}_-(x - \sigma t) & \text{if } x - \sigma t \in [d_0 - \eta, d_0 + \psi_\varepsilon(t)), \\ \underline{U}_+(x - \sigma t) & \text{if } x - \sigma t \in (d_0 + \psi_\varepsilon(t), d_0 + \eta], \\ \underline{U}(x - \sigma t) & \text{if } x - \sigma t \in \mathbb{R} \setminus (D \cup (d_0 - \eta, d_0 + \eta)), \end{cases}$$

where $\psi_\varepsilon \in \mathcal{C}^2([0, T_\varepsilon])$ is the solution with initial data $\psi_\varepsilon(0) = \varepsilon$ to the Rankine-Hugoniot condition

$$\sigma + \psi_\varepsilon'(t) = \Big(\frac{f(\underline{U}_+) - f(\underline{U}_-)}{\underline{U}_+ - \underline{U}_-}\Big)(d_0 + \psi_\varepsilon(t))$$

and $T_\varepsilon = \sup(\{t \geq 0 \ : \ \sup_{s \in [0,t]} |\psi_\varepsilon(s)| < \eta\}) > 0$. From the Rankine-Hugoniot condition we infer $|\psi_\varepsilon|(t) \geq |\varepsilon| e^{\alpha t}$. Hence $(u_{\varepsilon_n})_{n \in \mathbb{N}}$ for $(\varepsilon_n)_{n \in \mathbb{N}}$ nonzero, sufficiently small and converging towards zero satisfies the desired properties. □

3 Stability results

We now turn to the stability of constants, fronts, shocks and composite waves, under the spectral assumptions stated in the Theorem.

3.1 Stable equilibria

In this section, we discuss the spectral and asymptotic stability of constant states $\underline{u} \in \mathbb{R}$ with respect to regular perturbations under the condition

$$g(\underline{u}) = 0 \qquad \text{and} \qquad g'(\underline{u}) < 0\,. \tag{3.1}$$

Lemma 3.1 below shows in particular the spectral stability in $BUC^1(\mathbb{R})$ of constant states satisfying (3.1). Besides, it provides exponential decay estimates of regular solutions to

$$\partial_t v + a \partial_x v - b v = r \tag{3.2}$$

for any functions a close to $f'(\underline{u}) - \sigma$ and b close to $g'(\underline{u})$ in a suitable sense.

Lemma 3.1. *Assume a, $b \in BUC^0(\mathbb{R})$ with a bounded away from zero. Then the following holds.*

- *$L_{a,\,b} := -a\partial_x + b$ is a closed, densely-defined operator on $BUC^0(\mathbb{R})$ with domain $BUC^1(\mathbb{R})$.*
- *For any $\lambda \in \mathbb{C}$ such that*
$$\Re(\lambda) > \sup_{\mathbb{R}} b(\cdot)\,,$$
for any $\check{r} \in BUC^0(\mathbb{R})$, there exists a unique $\check{v}(\,\cdot\,; \lambda) \in BUC^1(\mathbb{R})$ such that
$$(\lambda - L_{a,\,b})\, \check{v}(\,\cdot\,; \lambda) \;=\; \check{r}$$
and moreover
$$\big\|\check{v}(\,\cdot\,; \lambda)\big\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\Re\lambda - \sup_{\mathbb{R}} b(\cdot)} \big\|\check{r}\big\|_{L^\infty(\mathbb{R})}\,.$$
Moreover if $\lambda \in \mathbb{R}$, $\lambda \in (\sup_{\mathbb{R}} b(\cdot), \infty)$ and $\check{r} \geq 0$, then $\check{v}(\,\cdot\,; \lambda) \geq 0$.

- *Assume moreover that $a \in BUC^1(\mathbb{R})$, b is constant,*

$$\Re(\lambda) > b - \inf_{\mathbb{R}} a'(\cdot)\,,$$

and $\check{r} \in W^{1,\infty}(\mathbb{R})$. Then

$$\big\|\partial_x \check{v}(\,\cdot\,;\lambda)\big\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\Re\lambda - b + \inf_{\mathbb{R}} a'(\cdot)}\big\|\partial_x \check{r}\big\|_{L^\infty(\mathbb{R})}\,.$$

Proof. Without loss of generality, we assume that a is positive. Let $\check{r} \in BUC^0(\mathbb{R})$. It is easy to check that when $\Re(\lambda) > \sup_{\mathbb{R}} b(\cdot)$, $\check{v}(\,\cdot\,;\lambda) \in BUC^1(\mathbb{R})$ is uniquely defined by

$$\check{v}(x;\lambda) := \int_{-\infty}^{x} e^{\int_y^x \frac{b(z)-\lambda}{a(z)}\,\mathrm{d}\,z}\,\frac{\check{r}(y)}{a(y)}\,\mathrm{d}\,y\,.$$

The second item is immediately deduced, in particular thanks to the chain inequalities

$$|\check{v}(x;\lambda)| \leq \frac{\big\|\check{r}\big\|_{L^\infty(\mathbb{R})}}{\Re\lambda - \sup_{\mathbb{R}} b(\cdot)}\int_{-\infty}^{x} e^{\int_y^x \frac{b(z)-\Re\lambda}{a(z)}\,\mathrm{d}\,z}\,\frac{\Re\lambda - b(y)}{a(y)}\,\mathrm{d}\,y = \frac{\big\|\check{r}\big\|_{L^\infty(\mathbb{R})}}{\Re\lambda - \sup_{\mathbb{R}} b(\cdot)}\,.$$

The third item is obtained in the same way after differentiation and integration by parts:

$$\begin{aligned}\partial_x \check{v}(x;\lambda) &= \frac{\check{r}(x)}{a(x)} + \int_{-\infty}^{x}\frac{b-\lambda}{a(x)}e^{\int_y^x \frac{b-\lambda}{a(z)}\,\mathrm{d}\,z}\,\frac{\check{r}(y)}{a(y)}\,\mathrm{d}\,y \;=\; \int_{-\infty}^{x} e^{\int_y^x \frac{b-\lambda}{a(z)}\,\mathrm{d}\,z}\,\frac{\partial_y\check{r}(y)}{a(x)}\,\mathrm{d}\,y\\ &= \int_{-\infty}^{x} e^{\int_y^x \frac{b-\lambda-a'(z)}{a(z)}\,\mathrm{d}\,z}\,\frac{\partial_y\check{r}(y)}{a(y)}\,\mathrm{d}\,y\,.\end{aligned}$$

This concludes the proof. □

We now turn to nonlinear stability results.

Proposition 3.2. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^{1+}(\mathbb{R})$ and $\underline{u} \in \mathbb{R}$ satisfying* (3.1)*. Then for any $C_0 > 1$, there exists $\varepsilon > 0$ such that for any $v_0 \in BUC^1(\mathbb{R})$ satisfying*

$$\big\|v_0\big\|_{W^{1,\infty}(\mathbb{R})} \leq \varepsilon\,,$$

the initial data $u(0,\cdot) = \underline{u} + v_0$ generates a global unique classical solution to (1.1)*, $u \in BUC^1(\mathbb{R}^+ \times \mathbb{R})$, and it satisfies for any $t \geq 0$*

$$\begin{aligned}\big\|u(t,\cdot) - \underline{u}\big\|_{L^\infty(\mathbb{R})} &\leq \big\|v_0\big\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,,\\ \big\|\partial_x u(t,\cdot)\big\|_{L^\infty(\mathbb{R})} &\leq \big\|\partial_x v_0\big\|_{L^\infty(\mathbb{R})} C_0\, e^{g'(\underline{u})\,t}\,.\end{aligned}$$

Using an additional concavity/convexity assumption, we may refine the above result by assuming only asymmetric initial smallness on the derivative of the initial data, consistently with the example of the introduction.

Proposition 3.3. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^{1+}(\mathbb{R})$ and $\underline{u} \in \mathbb{R}$ satisfying* (3.1) *and*

$$f''(\underline{u}) \neq 0.$$

Then for any $C_0 > 1$, there exists $\varepsilon > 0$ such that for any $v_0 \in BUC^1(\mathbb{R})$ satisfying

$$\big\|v_0\big\|_{L^\infty(\mathbb{R})} \leq \varepsilon \qquad \textit{and} \qquad \big\|(\mathrm{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\big\|_{L^\infty(\mathbb{R})} \leq \varepsilon\,,$$

the initial data $u(0,\cdot)=\underline{u}+v_0$ generates a global unique classical solution to (1.1)*, $u\in BUC^1(\mathbb{R}^+\times\mathbb{R})$, and it satisfies for any $t\geq 0$*

$$\begin{aligned}\big\|u(t,\cdot)-\underline{u}\big\|_{L^\infty(\mathbb{R})}&\leq \big\|v_0\big\|_{L^\infty(\mathbb{R})}C_0\,e^{g'(\underline{u})\,t}\,,\\ \big\|(\mathrm{sgn}(f''(\underline{u}))\,\partial_x u(t,\cdot))_-\big\|_{L^\infty(\mathbb{R})}&\leq \big\|(\mathrm{sgn}(f''(\underline{u}))\,\partial_x v_0)_-\big\|_{L^\infty(\mathbb{R})}C_0\,e^{g'(\underline{u})\,t}\,,\\ \big\|\partial_x u(t,\cdot)\big\|_{L^\infty(\mathbb{R})}&\leq \big\|\partial_x v_0\big\|_{L^\infty(\mathbb{R})}C_0\,e^{g'(\underline{u})\,t}\,.\end{aligned}$$

We sketch the proof of Proposition 3.3, the proof of Proposition 3.2 being similar.

Proof of Proposition 3.3. From Lemma 3.1, we may apply apply general theorems on evolution systems[4] and deduce from any $a\in\mathcal{C}^0([0,T),BUC^1(\mathbb{R}))$ and $b\in\mathcal{C}^0([0,T),BUC^0(\mathbb{R}))$ (with $T\in(0,\infty]$) an evolution system $\mathcal{S}_{a,b}$ on $BUC^0(\mathbb{R})$ generated by the family of operators $L_{a(t,\cdot),\,b(t,\cdot)}$, and such that for any $v_0\in BUC^0(\mathbb{R})$, any $0\leq s\leq t<T$

$$\big\|\mathcal{S}_{a,b}(s,t)\,v_0\big\|_{L^\infty(\mathbb{R})}\leq e^{\int_s^t\sup_{\mathbb{R}}b(\tau,\cdot)\,\mathrm{d}\,\tau}\big\|v_0\big\|_{L^\infty(\mathbb{R})}\,,$$

and $\mathcal{S}_{a,b}(s,t)\,v_0\geq 0$ if $v_0\geq 0$. If moreover b is constant, then $v_0\in BUC^1(\mathbb{R})$ yields for any $0\leq s\leq t<T$

$$\big\|\partial_x\mathcal{S}_{a,b}(s,t)\,v_0\big\|_{L^\infty(\mathbb{R})}\leq e^{(t-s)\,b-\int_s^t\inf_{\mathbb{R}}\partial_x a(\tau,\cdot)\,\mathrm{d}\,\tau}\big\|\partial_x v_0\big\|_{L^\infty(\mathbb{R})}\,.$$

Let $\varepsilon\in(0,1]$. Pick a classical solution $u=\underline{u}+v$ starting from $\underline{u}+v_0$ such that $\big\|v_0\big\|_{L^\infty(\mathbb{R})}\leq\varepsilon$. Then if u exists (as a classical solution) on $[0,t_0)$, for any $0\leq t<t_0$, we have the Duhamel formula

$$v(t,\cdot)\;=\;\mathcal{S}_{f'(\underline{u}+v),\,g'(\underline{u})}v_0+\int_0^t\mathcal{S}_{f'(\underline{u}+v),\,g'(\underline{u})}(s,t)\big(g(\underline{u}+v)-g(\underline{u})-g'(\underline{u})v\big)(s,\cdot)\,\mathrm{d}\,s\,.$$

Therefore if moreover for any $t\in[0,t_0)$, $\big\|v(t,\cdot)\big\|_{L^\infty(\mathbb{R})}\leq 2\varepsilon\,e^{g'(\underline{u})\,t}$, then for any $t\in[0,t_0)$

$$e^{-g'(\underline{u})\,t}\big\|v(t,\cdot)\big\|_{L^\infty(\mathbb{R})}\leq\big\|v_0\big\|_{L^\infty(\mathbb{R})}+\int_0^t\omega_g(2\varepsilon e^{g'(\underline{u})\,s})2\varepsilon e^{g'(\underline{u})\,s}\,\left(e^{-g'(\underline{u})\,s}\big\|v(s,\cdot)\big\|_{L^\infty(\mathbb{R})}\right)\mathrm{d}\,s$$

where $\omega_g(r)=\max_{|x-y|\leq r}\frac{|g'(x)-g'(y)|}{r}\in L^1_{\mathrm{loc}}(\mathbb{R})$, so that for any $t\in[0,t_0)$,

$$\big\|v(t,\cdot)\big\|_{L^\infty(\mathbb{R})}\leq\big\|v_0\big\|_{L^\infty(\mathbb{R})}\,e^{g'(\underline{u})\,t}\,e^{\frac{1}{|g'(\underline{u})|}\int_0^{2\varepsilon}\omega_g(r)\,\mathrm{d}\,r}.$$

Choosing ε sufficiently small, we may ensure $\exp(\frac{1}{|g'(\underline{u})|}\int_0^{2\varepsilon}\omega_g(r)\,\mathrm{d}\,r)<\min(\{2,C_0\})$ and a continuity argument yields that the L^∞ estimate of the Proposition holds as long as u persists as a classical solution. From the identity

$$\partial_x v(t,\cdot)=\mathcal{S}_{f'(\underline{u}+v),\,g'(\underline{u}+v)-f''(\underline{u}+v)\partial_x v}(0,t)\,\partial_x v_0\,,$$

by linearity and preservation of non negativity, we deduce

$$(\mathrm{sgn}(f''(\underline{u}))\partial_x v(t,\cdot))_-\leq\mathcal{S}_{f'(\underline{u}+v),\,g'(\underline{u}+v)-f''(\underline{u}+v)\partial_x v}(0,t)\,(\mathrm{sgn}(f''(\underline{u}))\partial_x v_0)_-\,.$$

Proceeding as above and lowering ε is necessary, we deduce the second estimate of the Proposition —again as long as u persists as a classical solution— and the third is obtained in the same way. This in particular rules out finite-time blow-up, and the proof is complete. ☐

[4]See for instance [12, Chapter 5, Theorem 3.1] with $X=BUC^0(\mathbb{R})$ and $Y=BUC^1(\mathbb{R})$, and apply [12, Chapter 5, Theorem 2.3] to reduce the verification of assumption (H_2) there to the case where b is constant.

3.2 Stable fronts

In this section we study the stability of bounded continuous fronts, that is $u : (t,x) \mapsto \underline{U}(x-\sigma t)$ solution to (1.1) with $f \in \mathcal{C}^2(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$, where $\sigma \in \mathbb{R}$ and $\underline{U} \in \mathcal{C}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is strictly monotonous. We assume the existence and uniqueness of a characteristic value:

$$\exists! u_\star \in \underline{U}(\mathbb{R}), \qquad f'(u_\star) = \sigma \quad \text{and} \quad g(u_\star) = 0\,, \tag{3.3}$$

and assume the non-degeneracy and (strict) spectral stability condition at the characteristic value:

$$f''(u_\star) \neq 0 \quad \text{and} \quad g'(u_\star) > 0\,. \tag{3.4}$$

Denoting $u_{\pm\infty} = \lim_{x\to\pm\infty} \underline{U}(x) \in \mathbb{R}$, we assume the non-degeneracy condition at infinity:

$$g(u_{\pm\infty}) = 0\ ,\ g'(u_{\pm\infty}) < 0 \quad \text{and} \quad f'(u_{\pm\infty}) \neq f'(u_\star)\,. \tag{3.5}$$

We denote $F : \underline{U}(\mathbb{R}) \to \mathbb{R}$ the $\mathcal{C}^0$ map defined by

$$\forall u \in \underline{U}(\mathbb{R}), \qquad F(u) = \begin{cases} \frac{g(u)}{f'(u)-\sigma} & \text{if } f'(u)-\sigma \neq 0, \\ \frac{g'(u)}{f''(u)} & \text{otherwise,} \end{cases}$$

so that

$$\forall x \in \mathbb{R}, \quad \underline{U}'(x) = F(\underline{U}(x))\,. \tag{3.6}$$

Linearizing (1.1) about the traveling solution u yields the linear equation

$$\partial_t v + f'(u)\partial_x v = \big(g'(u) - (f'(u))'\big)v, \qquad v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\,.$$

Studying the spectral stability of $\underline{U}$ as a solution to (1.1) hence amounts to studying the spectrum of the time-independent operator defined by

$$L_\star := -(f'(\underline{U}) - \sigma)\big(\partial_x - \frac{\underline{U}''}{\underline{U}'}\big).$$

However, as in Section 3.1, the nonlinear stability result will stem from resolvent estimates on a wider class of linear operators. Based on properties of $(f'(\underline{U}) - \sigma)$ —recall Proposition 1.6— we denote[5]

$$X^1_\star(\mathbb{R}) := \big\{ a \in BUC^1(\mathbb{R}) \ :\ a(0) = 0 \big\}$$

and consider $a \in X^1_\star(\mathbb{R})$ such that

$$\begin{cases} a(x) > 0 & \text{if } x > 0\,, \\ a(x) < 0 & \text{if } x < 0\,. \end{cases} \tag{3.7}$$

We then denote

$$L_a := -a\,\partial_x \ +\ a\,\frac{\underline{U}''}{\underline{U}'}$$

the closed, densely-defined operator on $BUC^1(\mathbb{R})$ with domain $BUC^2(\mathbb{R})$.

[5]Here we implicitly set the characteristic point to $x_\star = 0$, that is $\underline{U}(0) = u_\star$. This can be done without harm thanks to the translation invariance of the problem.

Lemma 3.4. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$ and $(\underline{U}, \sigma)$ satisfying* (3.3)-(3.4)-(3.5)-(3.6) *such that $F \in \mathcal{C}^1(\underline{U}(\mathbb{R}))$ and $\underline{U}(0) = u_\star$, and denote*

$$\theta := \min(\{g'(u_\star), -g'(u_{+\infty}), -g'(u_{-\infty})\}) > 0.$$

There exists $\chi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\inf_{\mathbb{R}} \Big((f'(\underline{U}))' - (f'(\underline{U}) - \sigma) \frac{\underline{U}''}{\underline{U}'} + (f'(\underline{U}) - \sigma)\, \chi \Big) \geq \theta.$$

For any $a \in X^1_\star(\mathbb{R})$ satisfying (3.7) *and*

$$\theta_a \ := \ \inf_{\mathbb{R}} \Big(a' - a \frac{\underline{U}''}{\underline{U}'} + a\, \chi \Big) > 0\,, \tag{3.8}$$

and for any $\lambda \in \mathbb{C}$ such that

$$\Re(\lambda) > -\theta_a\,,$$

for any $\check{r} \in X^1_\star(\mathbb{R})$, there exists a unique $\check{v}(\,\cdot\,;\lambda) \in BUC^2(\mathbb{R})$ such that

$$(\lambda - L_a)\, \check{v}(\,\cdot\,;\lambda) \ = \ \check{r}$$

and moreover $\check{v}(0) = 0$ and

$$\|\check{v}\|_{X^1_\star(\mathbb{R})} \ \leq \ \frac{1}{\Re(\lambda) + \theta_a} \|\check{r}\|_{X^1_\star(\mathbb{R})}$$

where $\|\cdot\|_{X^1_\star(\mathbb{R})}$ is a norm equivalent to $\|\cdot\|_{W^{1,\infty}(\mathbb{R})}$ on $X^1_\star(\mathbb{R})$ and is defined by

$$\forall v \in X^1_\star(\mathbb{R}), \qquad \|v\|_{X^1_\star(\mathbb{R})} \ := \ \big\| e^{-\int_0^\cdot \chi} \big[v' - \frac{\underline{U}''}{\underline{U}'} v \big] \big\|_{L^\infty(\mathbb{R})}\,. \tag{3.9}$$

Proof. Since $F \in \mathcal{C}^1$, $\underline{U} \in BUC^2(\mathbb{R})$, and one readily checks that

$$\chi(x) = \begin{cases} \max\big(\{\frac{\theta - (f'(\underline{U}))'(x) + (f'(\underline{U}(x)) - \sigma)\frac{\underline{U}''(x)}{\underline{U}'(x)}}{(f'(\underline{U}(x)) - \sigma)}, 0\}\big) & \text{when } x > 0, \\ \min\big(\{\frac{\theta - (f'(\underline{U}))'(x) + (f'(\underline{U}(x)) - \sigma)\frac{\underline{U}''(x)}{\underline{U}'(x)}}{(f'(\underline{U}(x)) - \sigma)}, 0\}\big) & \text{when } x < 0, \end{cases} \tag{3.10}$$

satisfies $\chi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as well as the desired inequality.

Dividing by $\underline{U}'$ the resolvent problem $(\lambda - L_a)\, \check{v} = \check{r}$, differentiating and then multiplying by $\underline{U}'$ yields

$$\Big(\lambda + a' - a\frac{\underline{U}''}{\underline{U}'}\Big) \Big(\frac{\check{v}}{\underline{U}'}\Big)' \underline{U}' + a \left(\Big(\frac{\check{v}}{\underline{U}'}\Big)' \underline{U}'\right)' = \Big(\frac{\check{r}}{\underline{U}'}\Big)' \underline{U}'.$$

Local solvability in $W^{1,\infty}(\mathbb{R})$ of the above yields, when $\Re(\lambda) > -\theta_a$,

$$\Big(\frac{\check{v}}{\underline{U}'}\Big)' \underline{U}'(x) \ = \ \int_0^x \frac{e^{-\int_y^x \frac{1}{a}(\lambda + a' - a\frac{\underline{U}''}{\underline{U}'})}}{a(y)} \Big(\frac{\check{r}}{\underline{U}'}\Big)' \underline{U}'(y)\, \mathrm{d}\, y\,, \tag{3.11}$$

and hence, using (3.7) and (3.8),

$$\big\| e^{-\int_0^\cdot \chi} \big[\check{v}' - \frac{\underline{U}''}{\underline{U}'} \check{v} \big] \big\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\Re(\lambda) + \theta_a} \big\| e^{-\int_0^\cdot \chi} \big[\check{r}' - \frac{\underline{U}''}{\underline{U}'} \check{r} \big] \big\|_{L^\infty(\mathbb{R})}.$$

When $\lambda \neq 0$, local solvability of the resolvent problem in $W^{1,\infty}(\mathbb{R})$ also enforces $\check{v}(0) = \check{r}(0)/\lambda$ and hence

$$\check{v}(x) \;=\; \frac{1}{\lambda}\frac{\check{r}(0)}{\underline{U}'(0)}\,\underline{U}'(x) + \underline{U}'(x)\int_0^x \frac{1}{\underline{U}'(y)}\,\big[\check{v}' - \frac{\underline{U}''}{\underline{U}'}\check{v}\big](y)\;\mathrm{d}\,y.$$

Hence the eigenvalue 0 of L_a is of multiplicity 1, with spectral projector defined by $(\Pi\check{r})(x) := \frac{\check{r}(0)}{\underline{U}'(0)}\,\underline{U}'(x)$. When $\check{r}(0) = 0$ the resolvent problem is uniquely solved in $W^{2,\infty}(\mathbb{R})$ for any $\Re(\lambda) > -\theta_a$ by

$$\check{v}(x) \;=\; \underline{U}'(x)\int_0^x \frac{1}{\underline{U}'(y)}\,\big[\check{v}' - \frac{\underline{U}''}{\underline{U}'}\check{v}\big](y)\;\mathrm{d}\,y$$

where we recall that $\check{v}' - \frac{\underline{U}''}{\underline{U}'}\check{v}$ has been uniquely determined in (3.11).

There remains to prove that $\|\cdot\|_{X^1_\star(\mathbb{R})}$ is equivalent to $\|\cdot\|_{W^{1,\infty}(\mathbb{R})}$ on $X^1_\star(\mathbb{R})$. Since $\chi \in L^1(\mathbb{R})$, the key argument consists in proving that, when $v \in X^1_\star(\mathbb{R})$, $\|v\|_{L^\infty}(\mathbb{R})$ is controlled up to a multiplicative constant by $\|\partial_x v - \frac{\underline{U}'}{\underline{U}'}v\|_{L^\infty(\mathbb{R})}$. This follows from the identity

$$v = \int_0^x \frac{\underline{U}'(x)}{\underline{U}'(y)}\big(\partial_x v - \frac{\underline{U}'}{\underline{U}'}v\big)(y)\;\mathrm{d}\,y$$

and the boundedness of $\underline{U}''/\underline{U}'$. □

We now turn to the consequence of Lemma 3.4 to the nonlinear stability of continuous fronts.

Proposition 3.5. *Let $f \in \mathcal{C}^{3+}(\mathbb{R})$, $g \in \mathcal{C}^{2+}(\mathbb{R})$ and $(\underline{U}, \sigma)$ satisfying (3.3)-(3.4)-(3.5)-(3.6) and such that $F \in \mathcal{C}^1(\underline{U}(\mathbb{R}))$, and denote*

$$\theta := \min(\{g'(u_\star), -g'(u_{+\infty}), -g'(u_{-\infty})\}) > 0.$$

For any $C_0 > 1$, there exists $\varepsilon > 0$ such that for any $v_0 \in BUC^1(\mathbb{R})$ satisfying

$$\|v_0\|_{W^{1,\infty}(\mathbb{R})} \leq \varepsilon, \tag{3.12}$$

and, denoting $x_\star \in \mathbb{R}$ the characteristic point such that $\underline{U}(x_\star) = u_\star$,

$$v_0(x_\star) = 0\,, \tag{3.13}$$

the initial data $u(0,\cdot) = \underline{U} + v_0$ generates a global unique classical solution to (1.1), $u \in BUC^1(\mathbb{R}^+ \times \mathbb{R})$, and it satisfies for any $t \geq 0$

$$\|u(t, \cdot + x_\star + \sigma t) - \underline{U}(\cdot + x_\star)\|_{X^1_\star(\mathbb{R})} \leq \|v_0\|_{X^1_\star(\mathbb{R})} C_0 e^{-\theta t}\,, \tag{3.14}$$

where $\|\cdot\|_{X^1_\star(\mathbb{R})}$ is defined in (3.9)-(3.10), and

$$u(t, x_\star + \sigma t) = \underline{U}(x_\star)\,. \tag{3.15}$$

Proof. Changing the reference frame, we may assume that $x_\star = 0$. We shall seek u under the form

$$u(t,x) = \underline{U}(x - \sigma t) + \widetilde{v}(t, x - \sigma t)$$

where $\widetilde{v}$ satisfies

$$\partial_t \widetilde{v} - L_{a(\widetilde{v})}\widetilde{v} \;=\; \mathcal{N}(\widetilde{v}), \tag{3.16}$$

denoting $a : X^1_\star(\mathbb{R}) \to X^1_\star(\mathbb{R})$ such that $a(\widetilde{v}) := f'(\underline{U} + \widetilde{v}) - \sigma$, and $\mathcal{N} : X^1_\star(\mathbb{R}) \to X^1_\star(\mathbb{R})$ such that

$$\mathcal{N}(\widetilde{v}) := (f'(\underline{U}) - f'(\underline{U} + \widetilde{v}))\,\frac{\underline{U}''}{\underline{U}'}\,\widetilde{v} + g(\underline{U} + \widetilde{v}) - g(\underline{U}) - g'(\underline{U})\widetilde{v} - (f'(\underline{U} + \widetilde{v}) - f'(\underline{U}) - f''(\underline{U})\widetilde{v})\,\underline{U}'.$$

One readily checks that there exists $\varepsilon_0 > 0$, depending only on f, $\underline{U}$ and χ —defined as in (3.10)— such that if u, the solution to (1.1) emerging from the initial data $u(0, \cdot) = \underline{U} + v_0$, persists as a classical solution to (1.1) on the time interval $[0, t_0)$ and satisfies for any $t \in [0, t_0)$, $\widetilde{v}(t, \cdot) \in X^1_\star(\mathbb{R})$ and

$$\left\|\widetilde{v}(t, \cdot)\right\|_{W^{1,\infty}(\mathbb{R})} \leq \varepsilon_0, \tag{3.17}$$

then $a(\widetilde{v}) \in \mathcal{C}([0, t_0); X^1_\star(\mathbb{R}))$ satisfies (3.7) and (3.8). Hence by the resolvent estimates obtained in Lemma 3.4 and the aforementioned general theorems on evolution systems, the family of operators $L_{a(t,\cdot)}$ generates an evolution system $\mathcal{S}_a$ on $X^1_\star(\mathbb{R})$ such that for any $0 \leq s \leq t < T$ and any $v_0 \in X^1_\star(\mathbb{R})$,

$$\left\|\mathcal{S}_a(s, t)\, v_0\right\|_{X^1_\star(\mathbb{R})} \leq e^{-\int_s^t \theta_{a(\tau,\cdot)}\,\mathrm{d}\,\tau}\left\|v_0\right\|_{X^1_\star(\mathbb{R})}. \tag{3.18}$$

Then, $\widetilde{v} \in \mathcal{C}([0, t_0); X^1_\star(\mathbb{R}))$ the classical solution to (3.16) satisfies Duhamel's formula

$$\widetilde{v}(t) = \mathcal{S}_{a(\widetilde{v})}(0, t)(\widetilde{v}_0) + \int_0^t \mathcal{S}_{a(\widetilde{v})}(s, t)\,\mathcal{N}(\widetilde{v}(s))\,\mathrm{d}\,s.$$

We may then proceed as in the proof of Proposition 3.3. Let $\varepsilon \in (0, \varepsilon_0)$. Assuming that for any $t \in [0, t_0)$, $\left\|\widetilde{v}\right\|_{X^1_\star(\mathbb{R})} \leq 2\varepsilon e^{-\theta t}$, we deduce from Duhamel's formula and the equivalence of $\left\|\cdot\right\|_{X^1_\star(\mathbb{R})}$ with $\left\|\cdot\right\|_{W^{1,\infty}(\mathbb{R})}$, a quantitative decay estimate on $\theta - \theta_a(t, \cdot)$ and sharp bounds on $\mathcal{N} : X^1_\star(\mathbb{R}) \to X^1_\star(\mathbb{R})$ which eventually yield

$$\left\|\widetilde{v}(t, \cdot)\right\|_{X^1_\star(\mathbb{R})} \leq C_\varepsilon \left\|v_0\right\|_{X^1_\star(\mathbb{R})} e^{-\theta t},$$

where $C_\varepsilon > 1$ depends only on f, g, $\underline{U}$, χ and ε, and can be brought arbitrarily close to 1 provided ε is sufficiently small. Choosing $\varepsilon > 0$ such that $C_\varepsilon < \min(\{2, C_0\})$ and such that from $\left\|\widetilde{v}\right\|_{X^1_\star(\mathbb{R})} \leq 2\varepsilon$ stems (3.17) —using again the equivalence of $\left\|\cdot\right\|_{X^1_\star(\mathbb{R})}$ with $\left\|\cdot\right\|_{W^{1,\infty}(\mathbb{R})}$— we infer from a continuity argument that (3.14) holds on $[0, t_0)$, and this in particular rules out finite-time blow up. The proof is complete. □

The assumption (3.13) is not a restriction to the orbital stability with asymptotic phase of stable fronts since, as stated below, any sufficiently small perturbation is an admissible perturbation of a shifted front. It is interesting to notice that the asymptotic phase is determined at initial time from the location of the characteristic point, contrarily to the more standard situation where the phase is only implicitly defined, as in the case of shocks described in the following section.

Lemma 3.6. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$ and $(\underline{U}, \sigma)$ satisfying* (3.3)-(3.4)-(3.6) *and such that $F \in \mathcal{C}^1(\underline{U}(\mathbb{R}))$, and denote $x_\star \in \mathbb{R}$ such that $\underline{U}(x_\star) = u_\star$. For any $C_0 > 1$, there exists $\varepsilon > 0$ such that for any $v_0 \in W^{1,\infty}(\mathbb{R})$ satisfying*

$$\left\|v_0\right\|_{W^{1,\infty}(\mathbb{R})} \leq \varepsilon,$$

there exists a unique $\tilde{x}_\star \in \mathbb{R}$ such that $\underline{U}(\tilde{x}_\star) + v_0(\tilde{x}_\star) = u_\star$, and it satisfies

$$|x_\star - \tilde{x}_\star| \leq \frac{C_0}{|\underline{U}'(0)|}\left\|v_0\right\|_{L^\infty(\mathbb{R})}.$$

Moreover, denoting $\underline{\tilde{U}} := \underline{U}(\cdot + x_\star - \tilde{x}_\star)$ and $\tilde{v}_0 := \underline{U} + v_0 - \underline{\tilde{U}}$, we have $\underline{\tilde{U}}(\tilde{x}_\star) = u_\star$, $\tilde{v}_0(\tilde{x}_\star) = 0$ and

$$\left\|\tilde{v}_0\right\|_{W^{1,\infty}(\mathbb{R})} \leq \left\|v_0\right\|_{W^{1,\infty}(\mathbb{R})} \;+\; \left\|\underline{U}'\right\|_{W^{1,\infty}(\mathbb{R})}|x_\star - \tilde{x}_\star|.$$

Proof. We use that $\underline{U}$ is strictly monotonic, $\underline{U}'(x_\star) \neq 0$ and $\underline{U} \in W^{2,\infty}(\mathbb{R})$. The existence of $\tilde{x}_\star$ is easily deduced from the intermediate value theorem, while uniqueness and the estimate on $|x_\star - \tilde{x}_\star|$ follows from the mean value theorem. The last estimate proceeds from the triangular and mean value inequalities. □

3.3 Stable shocks

In this section we show the orbital stability with asymptotic phase under regular perturbations of spectrally stable strictly entropy-admissible Riemann shocks of (1.1), that is

$$u(t,x) = \underline{U}(x - (d_0 + \sigma t))\,,$$

with initial shock position $d_0 \in \mathbb{R}$, speed $\sigma \in \mathbb{R}$ and wave profile $\underline{U}$

$$\underline{U}(x) = \begin{cases} \underline{u}_- & \text{if } x < 0\,, \\ \underline{u}_+ & \text{if } x > 0\,, \end{cases} \tag{3.19}$$

where $(\underline{u}_-, \underline{u}_+) \in \mathbb{R}^2$, $\underline{u}_+ \neq \underline{u}_-$ satisfy the equilibrium condition

$$g(\underline{u}_+) = 0 \quad \text{ and } \quad g(\underline{u}_-) = 0\,; \tag{3.20}$$

the speed $\sigma \in \mathbb{R}$ satisfies the Rankine-Hugoniot condition

$$f(\underline{u}_+) - f(\underline{u}_-) = \sigma(\underline{u}_+ - \underline{u}_-)\,, \tag{3.21}$$

and stability is ensured by the (strict) entropy admissibility

$$\begin{cases} \sigma \;>\; f'(\underline{u}_+)\,, \\ \frac{f(\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+) - f(\underline{u}_-)}{\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+ - \underline{u}_-} > \frac{f(\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+) - f(\underline{u}_+)}{\tau\,\underline{u}_- + (1-\tau)\,\underline{u}_+ - \underline{u}_+} & \text{for any } \;\tau \in (0,1)\,, \\ f'(\underline{u}_-) \;>\; \sigma\,, \end{cases} \tag{3.22}$$

and the spectral assumptions

$$g'(\underline{u}_+) < 0 \quad \text{ and } \quad g'(\underline{u}_-) < 0\,. \tag{3.23}$$

One could prove the nonlinear stability following the strategy of Section 3.1, after an analysis of the corresponding spectral problem, taking into account the position of the shock; recall Section 1.2. However it is more effective to rely directly on the result obtained for constant states, Proposition 3.2.[6]

Proposition 3.7. *Let $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^{1+}(\mathbb{R})$ and $(\sigma, \underline{u}_-, \underline{u}_+) \in \mathbb{R}^3$ satisfying* (3.20)-(3.21)-(3.22)-(3.23). *For any $C_0 > 1$, there exists $\varepsilon > 0$ and $C > 0$ such that for any $\psi_0 \in \mathbb{R}$ and $\widetilde{v}_0 \in BUC^1(\mathbb{R}^\star)$ satisfying*

$$\left\|\widetilde{v}_0\right\|_{W^{1,\infty}(\mathbb{R}^\star)} \leq \varepsilon\,, \tag{3.24}$$

there exists a unique global-in-time piecewise regular entropy solution to (1.1), *u, emerging from the initial data $u(0,\cdot) = (\underline{U} + \widetilde{v}_0)(\cdot + \psi_0)$ with $\underline{U}$ as in* (3.19). *Moreover, there exists $\psi \in \mathcal{C}^2(\mathbb{R}^+)$ satisfying $\psi(0) = 0$ and $u_\pm \in BUC^1(\mathbb{R}^+ \times \mathbb{R})$ such that for any $t \geq 0$*

$$u(t,x) = \begin{cases} u_-(t,x) & \textit{if } x < \psi_0 + \sigma t + \psi(t), \\ u_+(t,x) & \textit{if } x > \psi_0 + \sigma t + \psi(t), \end{cases} \tag{3.25}$$

and one has for any $t \geq 0$

$$\begin{aligned} \left\|u_\pm(t,\cdot) - \underline{u}_\pm\right\|_{L^\infty(\mathbb{R})} &\leq \left\|\widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\,t}\,, \\ \left\|\partial_x u_\pm(t,\cdot)\right\|_{L^\infty(\mathbb{R})} &\leq \left\|\partial_x \widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\pm)} C_0\, e^{g'(\underline{u}_\pm)\,t}\,, \\ |\psi(t) - \psi_\infty| + |\psi'(t)| &\leq \left\|\widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\star)} C\, e^{\max(\{g'(\underline{u}_+), g'(\underline{u}_-)\})\,t}\,, \end{aligned}$$

where $\psi_\infty = \lim_{t\to\infty} \psi(t)$ and satisfies $|\psi_\infty| \leq \left\|\widetilde{v}_0\right\|_{L^\infty(\mathbb{R}^\star)} C$.

[6] One could obtain in the same way a result based on Proposition 3.2, assuming only asymmetric smallness on the derivative of the perturbation, if one —or both— of the two end-states satisfies $f''(\underline{u}) \neq 0$.

Proof. We first extend the left- and right-components of the initial data so as to introduce $u_{0,\pm} \in BUC^1(\mathbb{R})$ such that

$$\forall x \in \mathbb{R}^{\pm}, \quad u_{0,\pm}(x) = (\underline{U} + \widetilde{v}_0)(x + \psi_0)$$

and

$$\|u_{0,\pm} - \underline{u}_{\pm}\|_{L^\infty(\mathbb{R})} \leq C_0^{1/2} \|\widetilde{v}_0\|_{L^\infty(\mathbb{R}^\pm)} \quad \text{and} \quad \|\partial_x u_{0,\pm}\|_{L^\infty(\mathbb{R})} \leq \|\partial_x \widetilde{v}_0\|_{L^\infty(\mathbb{R}^\pm)}.$$

By Proposition 3.2 with the amplification factor $C_0^{1/2}$ and provided the corresponding constraint on ε holds, we may define $u_\pm \in BUC^1(\mathbb{R}^+ \times \mathbb{R})$ as the global unique classical solutions to (1.1) emerging from the initial data $u_\pm(0,\cdot) = u_{0,\pm}$, and the desired estimates hold. The solution u is then obtained by patching together u_+ and u_- as in (3.25) where the discontinuity curve is defined through the Rankine-Hugoniot condition

$$(u_+ - u_-)(t, \psi_0 + \sigma t + \psi(t)) \times \big(\sigma + \psi'(t)\big) = \big(f(u_+) - f(u_-)\big)(t, \psi_0 + \sigma t + \psi(t)).$$

Existence and uniqueness of $\psi \in \mathcal{C}^2(\mathbb{R}^+)$ satisfying the above with initial datum $\psi(0) = 0$ follows from the standard theory on differential equations. The desired bounds on ψ are easily deduced from the corresponding bounds on $|u_\pm - \underline{u}_\pm|$, and $\psi_\infty = \int_0^\infty \psi'(t)\,\mathrm{d}t$. That (1.7) holds by lessening ε further if necessary follows from the continuity of

$$S_f : \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}, \qquad (a,b,\tau) \mapsto \frac{f(\tau\, a + (1-\tau)\, b) - f(a)}{\tau\, a + (1-\tau)\, b - a} - \frac{f(\tau\, a + (1-\tau)\, b) - f(b)}{\tau\, a + (1-\tau)\, b - b}.$$

Then u is an entropy solution to (1.1), and uniqueness is guaranteed by the theory due to Kružkov [6]. □

Remark 3.8. *It should be noted that $u_\pm$ are not defined uniquely, because we have freedom in the choice of the initial data $u_\pm(0,\cdot) = u_{0,\pm}$. However, the solution u obtained from* (3.25) *is of course unique in the class of entropy-admissible solutions.*

3.4 Stable composite waves

In this section we show the orbital stability with asymptotic phase under regular perturbations of bounded piecewise regular traveling wave solutions to (1.1) defined by $(\underline{U}, \sigma, D)$, under the following assumptions.

Hypothesis 3.9. *The set $D \subset \mathbb{R}$ is finite and non-empty. For any $d \in D$,* (1.3)-(1.4)-(1.5) *hold and*

$$\frac{[\, g(\underline{U})\,]_d}{[\,\underline{U}\,]_d} < 0\,.$$

For any connected component of $\mathbb{R} \setminus D$, J, one has either

i. *$\underline{U} \equiv \underline{u}$ is constant on J with $g(\underline{u}) = 0$, and $g'(\underline{u}) < 0$; or*

ii. *$\underline{U}$ is strictly monotonous on J, bounded and satisfies, for any $x \in J$, $\underline{U}'(x) = F(\underline{U}(x))$ where*

$$F(u) = \begin{cases} \frac{g(u)}{f'(u)-\sigma} & \text{if } f'(u) - \sigma \neq 0\,, \\ \frac{g'(u)}{f''(u)} & \text{otherwise,} \end{cases}$$

and $F \in \mathcal{C}^1(\underline{U}(J))$. Moreover, there exists a unique $u_\star \in \underline{U}(J)$ such that $f'(u_\star) = \sigma$, and one has $g(u_\star) = 0$, $f''(u_\star) \neq 0$ and $g'(u_\star) > 0$. We denote $x_\star \in \mathbb{R}$ the characteristic point, that is $\underline{U}(x_\star) = u_\star$.

We denote $u_{\pm\infty} = \lim_{x \to \pm\infty} \underline{U}(x)$, and one has $g(u_{\pm\infty}) = 0$ and $g'(u_{\pm\infty}) < 0$.

Recall that under non-degeneracy conditions on f and g, by Proposition 1.6, $\underline{U}$ may be constant only on unbounded connected components of $\mathbb{R}\setminus D$ and (see Remark 1.8) there exists at most —and hence exactly— one connected component on which $\underline{U}$ is strictly monotonous. For the sake of exposition, we provide the result only for the case where the number of elements in D is $|D|=2$; yet the equivalent statement with $|D|=1$ can be easily inferred.

As in Section 3.3, we shall infer our stability result by piecing together regular solutions emerging from extensions of the different components of the initial data. To this aim, it is convenient to introduce for the connected components $J_\star\subset\mathbb{R}\setminus D$ such that $\underline{U}$ is strictly monotonous, $\check{\underline{U}}$ the maximal solution to $\check{\underline{U}}'(x)=F(\check{\underline{U}}(x))$ such that $\check{\underline{U}}=\underline{U}$ on $J_\star$.

Proposition 3.10. *Let $f\in\mathcal{C}^{3+}(\mathbb{R})$, $g\in\mathcal{C}^{2+}(\mathbb{R})$ be non-degenerate and $(\underline{U},\sigma,D)$ satisfying Hypothesis 3.9 with $D=\{d_-,d_+\}$. For any $C_0>1$ there exists $\varepsilon>0$ and $C>0$ such that for any $D_0=\{d_{0,-},d_{0,+}\}\subset\mathbb{R}^2$ and $v_0\in BUC^1(\mathbb{R}\setminus D_0)$ satisfying*

$$\|v_0\|_{W^{1,\infty}(\mathbb{R}\setminus D_0)}+|d_{0,-}-d_-|+|d_{0,+}-d_+|\leq\varepsilon,$$

there exists a unique global piecewise regular entropy solution to (1.1)*, u, emerging from the initial data*

$$u(0,\cdot)=\begin{cases}u_{-\infty}+v_0(x) & \text{if } x<d_{0,-}\,,\\ \check{\underline{U}}(x)+v_0(x) & \text{if } d_{0,-}<x<d_{0,+}\,,\\ u_{+\infty}+v_0(x) & \text{if } x>d_{0,+}\,.\end{cases}$$

Moreover, there exist $\psi_\pm\in\mathcal{C}^2(\mathbb{R}^+)$ with $\psi_\pm(0)=0$ and $u_-,u_+,u_{(\star)}\in BUC^1(\mathbb{R}^+\times\mathbb{R})$ such that

$$\forall t\geq0,\qquad u(t,x)=\begin{cases}u_-(t,x) & \text{if } x<d_{0,-}+\psi_-(t)+\sigma t\,,\\ u_{(\star)}(t,x) & \text{if } d_{0,-}+\psi_-(t)<x-\sigma t<d_{0,+}+\psi_+(t)\,,\\ u_+(t,x) & \text{if } x>d_{0,+}+\psi_+(t)+\sigma t\,,\end{cases}\tag{3.26}$$

and for any $t\geq0$,

$$|\psi_\pm(t)-\psi_0|+|\psi_\pm'(t)|\leq\Big(\|v_0\|_{W^{1,\infty}(\mathbb{R}\setminus D_0)}+|d_{0,-}-d_-|+|d_{0,+}-d_+|\Big)\,C(1+t)e^{-\theta_\pm t}\,,$$

where $\theta_\pm=\min(\{-g'(u_{\pm\infty}),g'(u_\star),-\frac{[\,g(\underline{U})\,]_{d_\pm}}{[\,\underline{U}\,]_{d_\pm}}\})>0$ and $\psi_0\in\mathbb{R}$ is uniquely determined by

$$(\underline{U}+v_0)(x_\star+\psi_0)=u_\star=\underline{U}(x_\star)\,,$$

and satisfies $|\psi_0|\leq\frac{C_0}{|\underline{U}'(0)|}\|v_0\|_{L^\infty((d_{0,-},d_{0,+}))}$, and one has for any $t\geq0$,

$$u(t,x_\star+\psi_0+\sigma t)=u_{(\star)}(t,x_\star+\psi_0+\sigma t)=u_\star$$

and

$$\begin{aligned}\|u_\pm(t,\cdot)-u_{\pm\infty}\|_{L^\infty(\mathbb{R}^\pm)}&\leq\|v_0(\cdot+d_{0,\pm})\|_{L^\infty(\mathbb{R}^\pm)}C_0e^{g'(u_{\pm\infty})t},\\ \|(\partial_xu)(t,\cdot)\|_{L^\infty(\mathbb{R}^\pm)}&\leq\|(\partial_xv_0)(\cdot+d_{0,\pm})\|_{L^\infty(\mathbb{R}^\pm)}C_0e^{g'(u_{\pm\infty})t},\\ \|u_{(\star)}(t,\cdot+x_\star+\psi_0+\sigma t)-\check{\underline{U}}(\cdot+x_\star)\|_{X^1_\star(\tilde{J}^t_\star)}&\leq\|v_0\|_{X^1_\star(\tilde{J}^0_\star)}C_0e^{-g'(u_\star)t},\end{aligned}$$

where $\|\cdot\|_{X^1_\star(\tilde{J}_t)}$ is defined by (3.9)-(3.10)*, replacing θ therein with $g'(u_\star)$ and restricting the L^∞ norms to the domain $\tilde{J}^t_\star:=(d_{0,-}-x_\star-\psi_0+\psi_-(t),d_{0,+}-x_\star-\psi_0+\psi_+(t))$.*

Proof. We proceed as in the proof of Proposition 3.7, and the first step is to introduce regular extensions of the initial data for each connected component of $\mathbb{R} \setminus D_0$. On the connected components $J_\pm$ such that $u(0,\cdot) = u_{\pm\infty} + v_0$, we introduce $v_{0,\pm} \in BUC^1(\mathbb{R})$ such that $v_{0,\pm}(x) = v_0(x)$ for any $x \in J_\pm$, and

$$\left\| v_{0,\pm} \right\|_{L^\infty(\mathbb{R})} \leq C_0^{1/2} \left\| v_0 \right\|_{L^\infty(J_\pm)} \quad \text{and} \quad \left\| \partial_x v_{0,\pm} \right\|_{L^\infty(\mathbb{R})} \leq \left\| \partial_x v_0 \right\|_{L^\infty(J_\pm)}.$$

By Proposition 3.2 with the amplification factor $C_0^{1/2}$ and provided the corresponding constraint on ε holds, we obtain $u_\pm \in BUC^1(\mathbb{R}^+ \times \mathbb{R})$ global unique classical solutions to (1.1) emerging from the initial data $u_\pm(0,\cdot) = u_{\pm\infty} + v_{0,\pm}$, and the desired estimates hold.

On the connected components $J_\star$ such that $u(0,\cdot) = \underline{\check{U}} + v_0$, we first extend $\underline{\check{U}}$ to $\underline{\check{U}} \in BUC^1(\mathbb{R})$ by solving the differential equation $\underline{\check{U}}' = F(\underline{\check{U}})$ after modifying (if necessary) f and g on $\underline{\check{U}}(\mathbb{R} \setminus J_\star)$ in order to ensure that $f'(\underline{\check{U}}(x)) = f'(u_\star)$ if and only if $x = x_\star$, and $-g'(\lim_{x\to\pm\infty} \underline{\check{U}}(x)) > g'(u_\star)$. We then uniquely determine ψ_0 —by Lemma 3.6— as the solution to $(\underline{\check{U}} + v_0)(x_\star + \psi_0) = u_\star$, and set $\widetilde{v}_0 = v_0 + \underline{\check{U}} - \underline{\check{U}}(\cdot - \psi_0)$. Finally, we may define $\tilde{v}_{0,(\star)}$ so that $\tilde{v}_{0,(\star)}(\cdot + x_\star + \psi_0) \in X^1_\star(\mathbb{R})$, $\underline{\check{U}}(\cdot - \psi_0) + \tilde{v}_{0,(\star)} = u(0,\cdot)$ on $J_\star$ and

$$\left\| \tilde{v}_{0,(\star)}(\cdot + x_\star + \psi_0) \right\|_{X^1_\star(\mathbb{R})} \leq C_0^{1/2} \left\| \tilde{v}_0(\cdot + x_\star + \psi_0) \right\|_{X^1_\star(\tilde{J}^0_\star)}.$$

By Proposition 3.5 with the amplification factor $C_0^{1/2}$ and provided the corresponding constraint on ε holds, we obtain $u_{(\star)} \in BUC^1(\mathbb{R}^+ \times \mathbb{R})$ global unique classical solution to (1.1) emerging from the initial data $u_{(\star)}(0,\cdot) = \underline{\check{U}}(\cdot - \psi_0) + \tilde{v}_{0,(\star)}$, and satisfying the desired inequality.

We now construct u through (3.26), defining $\psi_\pm$ from the Rankine-Hugoniot condition at discontinuities:

$$\psi'_\pm(t) = F[u_{\mathrm{r},\pm}, u_{\mathrm{l},\pm}](t, \psi_\pm(t)) \quad \text{with} \quad F[u_\mathrm{r}, u_\mathrm{l}] := \frac{f(u_\mathrm{r}) - f(u_\mathrm{l})}{u_\mathrm{r} - u_\mathrm{l}}$$

where u_r (resp. u_l) is the limit from the right (resp. from the left) at the discontinuity, consistently with (3.26). Uniqueness and global existence of $\psi_\pm \in \mathcal{C}^2(\mathbb{R}^+)$ satisfying the desired estimates follows from standard results on differential equations, using the previously obtained estimates and the fact that

$$F[\underline{U}(\cdot), u_{-\infty}](d_-) = F[u_{+\infty}, \underline{U}(\cdot)](d_+) = \sigma\,,$$

$$\partial_x\big(F[\underline{U}(\cdot), u_{-\infty}]\big)(d_-) = \frac{[\,g(\underline{U})\,]_{d_-}}{[\,\underline{U}\,]_{d_-}} < 0 \quad \text{and} \quad \partial_x\big(F[u_{+\infty}, \underline{U}(\cdot)]\big)(d_+) = \frac{[\,g(\underline{U})\,]_{d_+}}{[\,\underline{U}\,]_{d_+}} < 0.$$

By construction, u is a global-in-time piecewise regular entropy solution to (1.1) provided Oleinik's (strict) entropy conditions hold on discontinuity curves, but this follows as in the proof of Proposition 3.7. □

4 Conclusion

In Section 3, we proved the nonlinear orbital stability with asymptotic phase of constant equilibria, fronts, shocks and composite wave solutions to (1.1) in $BUC^1(\mathbb{R}\setminus D)\times BUC^1(\mathbb{R})$ (where D is the set of discontinuities of the traveling wave) provided that the entropy and spectral stability assumptions stated in the Theorem hold. We have also proved in Section 2 the spectral and nonlinear instability of all other —under the non-degeneracy assumptions on f and g— bounded piecewise regular entropic traveling wave solutions to (1.1).

Further results using different functional spaces may be obtained. Firstly, the nonlinear asymptotic stability $BUC^k(\mathbb{R} \setminus D) \times BUC^k(\mathbb{R})$ for $k \geq 2$, without assuming smallness on higher derivatives, is easily deduced from the corresponding one when $k = 1$, after differentiating the equation. Furthermore, using the extension/patching strategy employed when dealing with discontinuities in Section 3.3 and 3.4, we may prove stability under perturbations admitting small strictly entropic discontinuities. Moreover, Proposition 3.3 allows to prove, by a classical approximation/compactness argument, the asymptotic stability of dissipative equilibria $\underline{u} \in \mathbb{R}$ such that $f''(\underline{u}) \neq 0$ (or strictly entropic shocks between such equilibria) in $BV(\mathbb{R})$ —the space of functions of bounded variation— allowing discontinuous initial data generating small rarefaction waves as well. We let the reader refer to [3, 4] for precise statements and comprehensive discussions.

References

[1] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[2] C. M. Dafermos. Generalized characteristics and the structure of solutions of hyperbolic conservation laws. *Indiana Univ. Math. J.*, 26(6):1097–1119, 1977.

[3] V. Duchêne and L. M. Rodrigues. Large-Time Asymptotic Stability of Riemann Shocks of Scalar Balance Laws. *SIAM J. Math. Anal.*, 52(1):792–820, 2020.

[4] V. Duchêne and L. M. Rodrigues. Stability in scalar balance laws: fronts and periodic waves. In preparation., 2020.

[5] H. T. Fan and J. K. Hale. Large time behavior in inhomogeneous conservation laws. *Arch. Rational Mech. Anal.*, 125(3):201–216, 1993.

[6] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[7] A. N. Lyberopoulos. Large-time structure of solutions of scalar conservation laws without convexity in the presence of a linear source field. *J. Differential Equations*, 99(2):342–380, 1992.

[8] A. N. Lyberopoulos. A Poincaré-Bendixson theorem for scalar balance laws. *Proc. Roy. Soc. Edinburgh Sect. A*, 124(3):589–607, 1994.

[9] C. Mascia. Continuity in finite time of entropy solutions for nonconvex conservation laws with reaction term. *Comm. Partial Differential Equations*, 23(5-6):913–931, 1998.

[10] C. Mascia. Qualitative behavior of conservation laws with reaction term and nonconvex flux. *Quart. Appl. Math.*, 58(4):739–761, 2000.

[11] C. Mascia and C. Sinestrari. The perturbed Riemann problem for a balance law. *Adv. Differential Equations*, 2(5):779–810, 1997.

[12] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[13] C. Sinestrari. Large time behaviour of solutions of balance laws with periodic initial data. *NoDEA Nonlinear Differential Equations Appl.*, 2(1):111–131, 1995.

[14] C. Sinestrari. Asymptotic profile of solutions of conservation laws with source. *Differential Integral Equations*, 9(3):499–525, 1996.

[15] C. Sinestrari. Instability of discontinuous traveling waves for hyperbolic balance laws. *J. Differential Equations*, 134(2):269–285, 1997.

[16] C. Sinestrari. The Riemann problem for an inhomogeneous conservation law without convexity. *SIAM J. Math. Anal.*, 28(1):109–135, 1997.