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Rigorous justification of the Favrie–Gavrilyuk approximation to the Serre–Green–Naghdi model

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Abstract

The (Serre–)Green–Naghdi system is a non-hydrostatic model for the propagation of surface gravity waves in the shallow-water regime. Recently, Favrie and Gavrilyuk proposed in Favrie and Gavrilyuk (2017 *Nonlinearity* **30** 2718–36) an efficient way of numerically computing approximate solutions to the Green–Naghdi system. The approximate solutions are obtained through solutions of an augmented quasilinear system of balance laws, depending on a parameter. In this work, we provide quantitative estimates showing that any sufficiently regular solution to the Green–Naghdi system is the limit of solutions to the Favrie–Gavrilyuk system as the parameter goes to infinity, provided the initial data of the additional unknowns is well-chosen. The problem is therefore a singular limit related to low Mach number limits with additional difficulties stemming from the fact that both order-zero and order-one singular components are involved.

Keywords: water-waves, shallow-water approximation, relaxation system, singular limit problem
Mathematics Subject Classification numbers: 35L60, 35Q35, 76B15, 76M45

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Motivation

The (Serre–)Green–Naghdi system arises as a model for the propagation of weakly dispersive surface gravity waves. It has been derived many times in the literature, and in particular in [3, 15, 24, 28–30]. More recently [19], it has been rigorously justified as an asymptotic model for the so-called water-waves system in the shallow-water regime. It can be seen as a second-order model refining the Saint-Venant system so as to take into account dispersive effects, and as such has received a fair amount of attention. Let us write down one of the many formulations of the Green–Naghdi system. Denoting $h(t,\mathbf{x})$ the depth of the water and $\mathbf{u}(t,\mathbf{x})$ the layer-averaged horizontal velocity at time $t\in\mathbb{R}$ and horizontal position $\mathbf{x}\in\mathbb{D}^d$ (where $\mathbb{D}=\mathbb{R}$ or $\mathbb{D}=\mathbb{T}$ and $d\in\{1,2\}$), the Green–Naghdi system in the flat-bottom situation reads

$$\begin{cases} \partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \\ \\ \partial_t \mathbf{u} + g\nabla h + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{3h}\nabla\big(h^2\ddot{h}\big) = \mathbf{0}, \end{cases} \tag{1.1}$$

where g is the gravitational acceleration and denoting $\dot{h}=\partial_t h+\mathbf{u}\cdot\nabla h$ and $\ddot{h}=\partial_t\dot{h}+\mathbf{u}\cdot\nabla\dot{h}$.

A difficulty arises when one tries to—numerically or analytically—solve the initial-value problem associated with (1.1) as, after using the equation of mass conservation to rewrite $\ddot{h}$, it is found necessary to invert the elliptic operator

$$\mathfrak{T}[h]:\mathbf{v}\mapsto h\mathbf{v}-\frac{1}{3}\nabla\big(h^3\nabla\cdot\mathbf{v}\big).$$

This is only a technical difficulty in the proof of the local well-posedness of the Cauchy problem [2, 9, 13, 22], but remains a severe issue for practical numerical simulations, as the cost of inverting this operator at each time step can be prohibitive, especially in dimension $d=2$. We refer to [21, 25] and references therein for several numerical schemes adapted to the Green–Naghdi system. The aforementioned issue is addressed in [10, 20], where the authors introduce a new class of models which enjoy the same precision as the original Green–Naghdi system—as an asymptotic model for the water-waves system—but for which the elliptic operator playing the role of $\mathfrak{T}[h]$ is independent of time. A different direction of investigation is proposed in the recent paper by Favrie and Gavrilyuk [11]. By modifying the lagrangian associated with the variational formulation of the Green–Naghdi system, the authors derive a system of balance laws depending on an augmented set of unknowns and on a free parameter:

$$\begin{cases} \partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + g\nabla h + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\lambda}{3h}\nabla\big(\frac{\eta}{h}(\eta-h)\big) = \mathbf{0}, \\ \partial_t\eta + \mathbf{u}\cdot\nabla\eta = w, \\ \partial_t w + \mathbf{u}\cdot\nabla w = -\frac{\lambda}{h^2}\big(\eta-h\big). \end{cases} \tag{1.2}$$

The claim is that in the limit $\lambda\to\infty$, solutions to (1.2) approach solutions to (1.1). Indeed we expect, using the fourth and third equations of (1.2):

$$\eta = h+\mathcal{O}(\lambda^{-1}) \quad\text{ and }\quad \lambda\big(\eta-h\big) = -h^2\ddot{\eta} = -h^2\ddot{h}+\mathcal{O}(\lambda^{-1}),$$

and we recover (1.1) when plugging the truncated approximations in the second equation of (1.2).

The aim of this work is to produce quantitative estimates which allow to rigorously prove that the Favrie–Gavrilyuk system (1.2) produces arbitrarily precise approximate solutions to the Green–Naghdi system (1.1) on the relevant timescale.

Among other things, our work gives insights to how large λ should be chosen and how initial data for η and w should be set in order for the corresponding solution to (1.2) to be a valid approximation to the solution of the Green–Naghdi system (1.1); and hence to surface gravity waves in the shallow-water regime.

1.2. Main results

The main ingredient in the proof of theorem 1.3 below will be *a priori* estimates, which need to be assessed uniformly with respect to the parameter λ. In order to provide useful results, we also need to provide estimates which are uniform with respect to the other parameters of the system, and in particular the (small) shallowness parameter, which measures the precision of the Green–Naghdi system.

Hence for proper comparison, we start by non-dimensionalizing the systems (1.1) and (1.2). A natural choice of scaling in the shallow-water regime[1] yields respectively

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\mathbf{u}) = 0, \\ \\ \partial_t\mathbf{u} + \nabla\zeta + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\mu}{3h}\nabla\big(h^2\ddot{h}\big) = \mathbf{0}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t\mathbf{u} + \nabla\zeta + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\lambda\mu}{3h}\nabla\big(\frac{\eta}{h}(\eta-h)\big) = \mathbf{0}, \\ \partial_t\eta + \mathbf{u}\cdot\nabla\eta = w, \\ \partial_t w + \mathbf{u}\cdot\nabla w = -\frac{\lambda}{h^2}\big(\eta-h\big). \end{cases} \tag{1.4}$$

Above, ζ is the dimensionless surface deformation and we will always denote $h = 1+\zeta$. The shallowness parameter μ is the square of the ratio of the typical depth of the layer to the typical horizontal wavelength of the wave, and is assumed to be small in the shallow-water regime: roughly speaking, regular solutions to the Green–Naghdi system (1.3) approximate corresponding solutions to the so-called water-waves system up to an error of size $\mathcal{O}(\mu^2 t)$ on the 'quadratic' time scale i.e. up to a maximal time inversely proportional to the size of the initial data; see [19]. We aim at proving that solutions to system (1.4) when λ is large and initial data for (η, w) are well-prepared approach solutions to the Green–Naghdi system (1.3), uniformly with respect to the parameter μ and on the quadratic time scale.

We give a precise statement of our main results below. We let the reader refer to section 2.1 for a description of the notations involved.

[1] We use the scaled variables

$$\mathbf{x} \leftarrow \mathbf{x}/L \quad ; \quad t \leftarrow t \times \sqrt{gH}/L,$$

and scaled unknowns

$$\mathbf{u} \leftarrow \mathbf{u}/\sqrt{gH} \quad ; \quad h \leftarrow h/H \quad ; \quad \zeta \leftarrow \zeta/H.$$

The choice is less obvious for the augmented unknowns which have no direct physical interpretation. In view of their dimension and the expected behaviour as $\lambda \to \infty$, we set

$$w \leftarrow w/\sqrt{gH} \times (L/H) \quad ; \quad \eta \leftarrow \eta/H.$$

Thus we scale w differently from $\mathbf{u}$, because the former represents typically a vertical velocity while the latter is the layer-averaged horizontal velocity. As for λ, we set

$$\lambda \leftarrow \lambda/(gH) \times (L/H)^2.$$

Here again this choice reflects the fact that λ compares with a vertical acceleration, times a vertical length. We denote $\mu \stackrel{\text{def}}{=} (H/L)^2$.

Because system (1.4) is a symmetrizable hyperbolic quasilinear system (as it is checked in section 2.3 below), the well-posedness of the corresponding initial-value problem is provided by standard theory; see e.g. [5].

Theorem 1.1. *Let $s \in \mathbb{R}$ with $s > 1 + d/2$. Then for any $\lambda, \mu \in (0, \infty)$ and any $U_0 = (\zeta_0, \mathbf{u}_0, \eta_0 - 1, w_0) \in H^s(\mathbb{D}^d)^{d+3}$ satisfying the hyperbolicity condition*

$$h_0 \stackrel{\text{def}}{=} 1 + \zeta_0 \geqslant h_\star > 0, \tag{1.5}$$

there exists a unique maximal strong solution $U = (\zeta, \mathbf{u}, \eta - 1, w) \in \mathcal{C}^0([0, T^\star); H^s(\mathbb{D}^d))^{d+3}$ to (1.4) with $U\big|_{t=0} = U_0$, where $T^\star > 0$ is the maximal time of existence. Moreover, one has $U \in \bigcap_{j=0}^{\lfloor s \rfloor} \mathcal{C}^j([0, T^\star); H^{s-j}(\mathbb{D}^d)^{d+3})$, and either $T^\star = \infty$ or $\lim_{t \nearrow T^\star} \big|U\big|_{W^{1,\infty}}(t) = \infty$.

Solutions to the Favrie–Gavrilyuk system are valuable approximations to the Green–Naghdi system (in the sense of consistency) as long as several space and time derivatives of the solutions are uniformly bounded, as stated below.

Theorem 1.2. *Let $s \geqslant 2$ and $M^\star > 0$. There exists $C > 0$ such that for any $\lambda, \mu \in (0, \infty)$, $T > 0$ and $U = (\zeta, \mathbf{u}, \eta - 1, w) \in \bigcap_{j=0}^{2} \mathcal{C}^j([0, T); H^{s-j}(\mathbb{D}^d))^{d+3}$ strong solution to (1.4) satisfying $h \stackrel{\text{def}}{=} 1 + \zeta > 0$ and*

$$\sup_{t \in [0,T)} \left(\big|U\big|_{H^s} + \big|\partial_t U\big|_{H^{s-1}} + \big|\partial_t^2 U\big|_{H^{s-2}} \right) \leqslant M^\star,$$

one has

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla \zeta + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\mu}{3h} \nabla \left(h^2 \ddot{h} \right) = \frac{\mu}{3h} \nabla r, \end{cases} \tag{1.6}$$

with $r = -h\ddot{\eta}(\eta - h) - h^2(\ddot{\eta} - \ddot{h}) \in \mathcal{C}^0([0, T^\star); H^{s-2}(\mathbb{D}^d))$ and if one has additionally $\partial_t^j w \in L^1(0, T; H^{s+1-j}(\mathbb{D}^d))$ for any $j \in \{0, \ldots, 3\}$, then

$$\big|r\big|_{H^{s-2}} \leqslant \lambda^{-1} C \sum_{j=0}^{3} \big|\partial_t^j w\big|_{H^{s+1-j}}.$$

Proof. The formula for r comes from straightforward manipulations on (1.4). By lemma 2.2 we infer $r \in \mathcal{C}^0([0, T^\star); H^{s-2}(\mathbb{D}^d))$ and

$$\forall t \in [0, T^\star), \qquad \big|r\big|_{H^{s-2}}(t) \leqslant C(M^\star) \sum_{j=0}^{2} \big|\partial_t^j (\eta - h)\big|_{H^{s-j}}(t).$$

The desired estimate is deduced, applying lemma 2.2 to the last equation of (1.4). □

Of course, there is no reason to hope *a priori* that the maximal solutions to (1.4) with initial data in a given ball of $H^s(\mathbb{D}^d)^{d+3}$—or continuously embedded normed spaces—satisfy the estimates of theorem 1.2 on a relevant time scale uniformly with respect to the parameters λ (large) and μ (small). The main result of this work is to prove that it is possible to *prepare the initial data* (for η and w) so that such property holds.

All our results from now on are restricted to the following set of parameters

$$\mathcal{S}_\nu = \{(\lambda,\mu)\in(0,\infty)^2,\ \lambda^{-1}+\mu+(\lambda\mu)^{-1}\leqslant \nu\}, \tag{1.7}$$

where ν should be thought as a prescribed constant of order of magnitude one. The results are valid for any choice of $\nu>0$, but of course not uniformly as $\nu\to\infty$. The first two restrictions in $\mathcal{S}_\nu$ are harmless in our framework, but the second one already hints at a possibly non-uniform behaviour with respect to small values of μ. Let us further prepare the Favrie–Gavrilyuk system through a change of variables which allows to balance the singular terms in (1.4). Introducing

$$\iota \stackrel{\rm def}{=} (\mu\lambda)^{1/2}(\eta-h) \quad ; \quad \kappa \stackrel{\rm def}{=} \mu^{1/2}h^{-1}w, \tag{1.8}$$

we see that (1.4) is equivalent to

$$\begin{cases} \partial_t\zeta+\nabla\cdot(h\mathbf{u})=0,\\ \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\zeta-\frac{1}{3h}\nabla\Big((\mu\lambda)^{1/2}\iota+\frac{\iota^2}{h}\Big)=\mathbf{0},\\ \partial_t\iota+\mathbf{u}\cdot\nabla\iota=\lambda^{1/2}(h\kappa+\mu^{1/2}h\nabla\cdot\mathbf{u}),\\ \partial_t(h\kappa)+\mathbf{u}\cdot\nabla(h\kappa)=-\lambda^{1/2}h^{-2}\iota. \end{cases} \tag{1.9}$$

The following result shows that one can control solutions to (1.9) on a time interval uniform with respect to λ (sufficiently large) and μ provided that the initial data is well-prepared.

Theorem 1.3. *Let $m,s\in\mathbb{N}$ with $s>1+d/2$, $1\leqslant m\leqslant s$, and $h_\star, M_0^\star,\nu>0$. Set also $\delta_\star\in(0,1)$ if $m=s$. There exist $\nu_\star, T, C_0>0$ such that for any $(\lambda,\mu)\in\mathcal{S}_\nu$ satisfying $\lambda\mu\geqslant\nu_\star$, for any $\tilde\lambda\in[1,\lambda\mu]$ and for any maximal strong solution $V=(\zeta,\mathbf{u},\iota,\kappa)\in C^0([0,T^\star);H^s(\mathbb{D}^d))^{d+3}$ to system (1.9) such that $h|_{t=0}=1+\zeta|_{t=0}\geqslant h_\star$ (provided by theorem 1.1) and satisfying additionally*

$$M_0 \stackrel{\rm def}{=} \sum_{j=0}^m \big|\partial_t^j V\big|_{H^{s-j}}(0)+\sum_{j=m+1}^s \tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j V\big|_{H^{s-j}}(0)\leqslant M_0^\star, \tag{1.10}$$

one has $T^\star>(M_0T)^{-1}$ and for any $t\in[0,(M_0T)^{-1}]$,

$$\sum_{j=0}^m \big|\partial_t^j V\big|_{H^{s-j}}(t)+\sum_{j=m+1}^s \tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j V\big|_{H^{s-j}}(t)\leqslant C_0M_0.$$

If $m=s$, we can withdraw the condition $\lambda\mu\geqslant\nu_\star$ and replace it with the sharper

$$(1-\delta_\star)(\lambda\mu)^{1/2}\geqslant \max\Big\{|(\kappa h)|_{t=0}|,\frac12|(\iota h^{-1})|_{t=0}|\Big\}.$$

It is important to notice that the above result holds with any $\tilde\lambda\in[1,\lambda\mu]$ but not with $\tilde\lambda=\lambda$ uniformly with respect to μ small. If it were the case, then the initial assumption on the high-order time derivatives of the unknown ($j\geqslant m+1$ in (1.10)) would be irrelevant as, using the system of equation (1.9) and product estimates (see lemma 2.2 below), we can estimate high-order time derivatives of V from lower-order time derivatives, with a cost of powers of $\lambda^{1/2}$:

$$\sum_{j=m+1}^s \lambda^{\frac{m-j}{2}}\big|\partial_t^j V\big|_{H^{s-j}}\leqslant C\Big(h_\star,\nu,\sum_{j=0}^m\big|\partial_t^j V\big|_{H^{s-j}}\Big)\times\sum_{j=0}^m\big|\partial_t^j V\big|_{H^{s-j}}.$$

In particular, in the strong dispersion regime ($\mu \approx 1$), the explicit condition

$$\big|V\,|_{t=0}\,\big|_{H^s} + \lambda^{1/2}\big|\iota\,|_{t=0}\,\big|_{H^{s-1}} + \lambda^{1/2}\big|\kappa\,|_{t=0}\, + \mu^{1/2}\nabla\cdot\mathbf{u}\,|_{t=0}\,\big|_{H^{s-1}} \leqslant M_0^\star$$

is sufficient (applying theorem 1.3 with $\tilde{\lambda} = \lambda\mu \approx \lambda$ and $m = 1$) to guarantee the existence and uniform control of the corresponding solution—but not its time derivatives—on a time interval uniform with respect to λ sufficiently large. In the weak dispersion or shallow-water regime ($\mu \ll 1$), the assumption (1.10) in theorem 1.3 is a strong constraint on the initial behavior of the solution, and it is natural to ask whether it is possible, for a given initial physical state defined by $\zeta\,|_{t=0}\,, \mathbf{u}\,|_{t=0}$ and $\mu > 0$, to provide initial data for the additional components $\eta\,|_{t=0}$ and $w\,|_{t=0}$ such that the corresponding solution to (1.4) satisfies (1.10) uniformly with respect to large λ and small μ. We answer positively below.

Theorem 1.4. *Let $s, m \in \mathbb{N}$, $s > d/2 + 1$, $s \geqslant m + 1$ and $h_\star, M_0^\star, \nu > 0$. There exists $C_m, C'_m > 0$ such that for any $(\lambda, \mu) \in \mathcal{S}_\nu$ and any $(\zeta_0, \mathbf{u}_0) \in H^s(\mathbb{D}^d)^{1+d}$ such that $h_0 = 1 + \zeta_0 \geqslant h_\star > 0$ and*

$$M_0 \stackrel{\rm def}{=} \big|\zeta_0\big|_{H^s} + \big|\mathbf{u}_0\big|_{H^s} \leqslant M_0^\star,$$

the following holds. There exists $c^{(j)} \in H^s(\mathbb{D}^d)$ for $j \in \{1, \cdots, m\}$ such that the strong solution to (1.4) with initial data $U^{(m)}\,|_{t=0} = (\zeta_0, \mathbf{u}_0, \eta_0^{(m)}, w_0^{(m)})$ where

$$w_0^{(m)} = \sum_{\substack{j \text{ odd} \\ 1 \leqslant j \leqslant m}} \lambda^{-(j-1)/2} c^{(j)} \quad \text{and} \quad \eta_0^{(m)} = h_0 + \sum_{\substack{j \text{ even} \\ 2 \leqslant j \leqslant m}} \lambda^{-j/2} c^{(j)} \tag{1.11}$$

satisfies

$$\sum_{j=0}^{m+1} \big|\partial_t^j U^{(m)}\big|_{H^{s-j}}(0) + \lambda \sum_{j=0}^{m} \big|\partial_t^j(\eta^{(m)} - h^{(m)})\big|_{H^{s-j}}(0) \leqslant C_m M_0. \tag{1.12}$$

Moreover, we have for any $j \in \{1, \cdots, m\}$

$$\begin{cases} \big|c^{(j)}\big|_{H^{s-j}} + \mu^{j/2}\big|c^{(j)}\big|_{H^s} \leqslant C'_m M_0, & \text{if } j \text{ is even,} \\ \big|c^{(j)}\big|_{H^{s-j}} + \mu^{(j-1)/2}\big|c^{(j)}\big|_{H^{s-1}} \leqslant C'_m M_0, & \text{if } j \text{ is odd.} \end{cases} \tag{1.13}$$

We can choose $c^{(1)} = -h_0 \nabla\cdot\mathbf{u}_0$ and $c^{(2)}$ the unique solution to

$$\mathfrak{t}[h_0]c^{(2)} = \mathbf{u}_0\cdot\nabla(\nabla\cdot\mathbf{u}_0) - (\nabla\cdot\mathbf{u}_0)^2 - \Delta\zeta_0 - \nabla\cdot((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0)$$

where we define

$$\mathfrak{t}[h]\varphi \stackrel{\rm def}{=} h^{-3}\varphi - \frac{\mu}{3}\nabla\cdot\left(h^{-1}\nabla\varphi\right).$$

Remark 1.5. The expression for $c^{(2)}$ emerges when solving

$$(h^2\ddot{\eta} - h^2\ddot{h})\,|_{t=0}\, = -\lambda(\eta - h)\,|_{t=0}\, - (h^2\ddot{h})\,|_{t=0}\, = \mathcal{O}(\lambda^{-1}).$$

The operator $\mathfrak{t}[h]$ is one-to-one and onto (see lemma 2.3 below) if $\inf h > 0$ and is in some sense conjugate to $\mathfrak{T}$ defined above, as one has for any sufficiently regular $(h, \varphi, \mathbf{u})$

$$\mathfrak{T}[h](h^{-1}\nabla\varphi) = \nabla(h^3\mathfrak{t}[h]\varphi) \quad \text{and} \quad \nabla\cdot(h^{-1}\mathfrak{T}[h]\mathbf{u}) = \mathfrak{t}[h](h^3\nabla\cdot\mathbf{u}).$$

Remark 1.6. A direct application of theorems 1.3 and 1.4 shows that for any sufficiently regular initial data $(\zeta_0, \mathbf{u}_0)$ satisfying the non-cavitation assumption (1.5), one may associate a solution to (1.4) satisfying the estimates of theorem 1.2 uniformly with respect to μ possibly small and λ sufficiently large, on the quadratic time scale (i.e. inversely proportional to the size of the initial data). Henceforth we produce $(\zeta, \mathbf{u})$ satisfying the Green–Naghdi system up to a residual of size $\mathcal{O}(\lambda^{-1}\mu)$, i.e. approximate solutions in the sense of consistency. Using energy estimates on the linearized Green–Naghdi system (see [2, 9, 13, 22]), we deduce that the difference between such solution and the exact solution to the Green–Naghdi system (1.3) with the same initial data is of size $\mathcal{O}(\lambda^{-1}\mu t)$ on the quadratic time scale. This should be compared with the results of the previously mentioned works (and references therein) showing that the solution to the Green–Naghdi system is at a distance $\mathcal{O}(\mu^2 t)$ to the solution of the full water-waves system with corresponding initial data on the same time scale. Hence the Favrie–Gavrilyuk system produces as precise approximate solutions for long gravity waves as the Green–Naghdi system itself as soon as $\lambda \gtrsim \mu^{-1}$ and the initial data for (η, w) is suitably chosen.

1.3. Outline

The remainder of this work is organized as follows. In section 1.4 we describe and comment our results and strategy in the light of relevant references in the literature of singular limit problems.

In section 2.1, we describe our notations. Section 2.2 contains technical tools such as product estimates in Sobolev spaces and an elliptic estimate on the operator $\mathfrak{t}[h]$. We show in section 2.3 that the Favrie–Gavrilyuk system is hyperbolic under the usual non-cavitation assumption. We exhibit in section 2.4 the symmetric structure of the system upon which our results are based.

Section 3 contains the proof of theorem 1.3. Section 4 contains the proof of theorem 1.4. Finally, section 5 is dedicated to a summary and concluding remarks.

1.4. Strategy

As aforementioned, the main tool for proving the above results are *a priori* estimates, which should hold uniformly with respect to the parameters $\lambda, \mu \in \mathcal{S}_\nu$. In order to obtain these estimates, we make use of a symmetric structure which is fairly easily deduced from the formulation (1.9). As a matter of fact, we show in section 2.4 below that the system can be written (when $d = 2$) as

$$\mathsf{S}_t(V)\partial_t V + \mathsf{S}_x(V)\partial_x V + \mathsf{S}_y(V)\partial_y V = \lambda^{1/2}\mathsf{J}^\mu V + G(V),$$

where $\mathsf{S}_t, \mathsf{S}_x, \mathsf{S}_y$ are smooth functions of V with values into symmetric matrices, J^μ is a skew-symmetric constant-coefficient differential operator, and G is a smooth function. Moreover S_t is positive definite in a hyperbolicity domain containing a neighborhood of the origin.

We are obviously looking at a *singular limit* problem. Such problems, and in particular incompressible or low Mach number limits in the context of fluid mechanics, have a very rich history, which we shall not recall. We will only let the interested reader refer to, e.g. [1, 14, 27] for comprehensive reviews. Due to the non-trivial symmetrizer in front of the time derivative, the linearized system does not appear to be uniformly well-posed in Sobolev spaces as $\lambda \to \infty$ since small perturbations of the initial data might cause large changes in solutions. This is a noteworthy feature of the incompressible limit of the non-isentropic Euler equations, as studied in particular in [23]. However, our problem is different in nature as we do not wish to deal with large oscillations in time but rather aim at discarding them as spurious products of the approximation procedure. Hence we willingly restrict our study to *well-prepared initial data*, and as such our work is more directly related to pioneering works of Browning and Kreiss [7], Klainerman and Majda [18], and Schochet [26]. In fact our proof of theorem 1.3 closely follows the one of [26]; while the proof of theorem 1.4 is strongly inspired by [7]. However in both cases the proof requires significant adaptations in order to take into account the fact that the singular operator, J^μ*, is not homogeneous of order one*.

The most serious novel difficulty stems from the fact that the contribution from order-zero terms in J^μ are less well-behaved than order-one contributions, and that the latter are multiplied by a vanishing prefactor as $\mu \to 0$. This is the reason for the shortcoming described below theorem 1.3. We would like to explain now this discrepancy with the more standard setting—studied in the previously mentioned references—where J^μ is homogeneous of order one. A toy model for the latter situation could be the following:

$$\partial_t u = \frac{1}{\epsilon} h \partial_x u \quad ; \quad \partial_t h = 0.$$

Here u is the singular variable while h is a regular variable, given and independent of time. The problem is reduced to a linear problem with variable coefficients, which is readily solvable by the methods of characteristics if we assume for instance that h, u are initially regular and for any $x \in \mathbb{R}$, $0 < h_\star \leqslant h(x) \leqslant h^\star < \infty$. We see that variations of size δ in h produce variations of size 1 on u at time $t = \epsilon/\delta$. However, the solution and its space derivatives remain controlled for all times, uniformly with respect to ϵ small. This behavior is not shared for the toy model corresponding to J^μ homogeneous of order zero, namely

$$\partial_t u = \mathrm{i} \frac{1}{\epsilon} h u \quad ; \quad \partial_t h = 0.$$

The problem is now an ordinary differential equation in time where the space variable is a parameter. The solution $u(t,x) = u_0(x) \exp(\mathrm{i} t h(x)/\epsilon)$ strongly oscillates with a different rate as $h(x)$ takes different values. Hence for positive times, the solution exhibits small scale oscillations, and space derivatives are not uniformly controlled with respect to the parameter ϵ small. If variations of h are of size δ, it is necessary to prepare the initial data $u\,|_{t=0} = \mathcal{O}(\epsilon^m)$ in order to control m space derivatives of the solution at time $t = 1/\delta$. Our situation is roughly speaking a combination of the above where the size of μ measures the relative strength of the two influences. Based on the necessary properties satisfied by J^μ (in particular lemma 3.4 below) a toy model could be

$$\partial_t u = \mathrm{i} \frac{1}{\epsilon} h (1 - \mu \partial_x^2)^{\frac{1}{2}} u \quad ; \quad \partial_t h = 0.$$

Consistently with theorem 1.3—and following the lines of its proof—the assumption $u\,|_{t=0} = \mathcal{O}(\mu^{m/2})$ is sufficient to control m space derivatives of the solution at time $t = 1/\delta$, uniformly with respect to ϵ small. We let the reader refer to section 3 for a more detailed discussion on the strategy developped in [7] and [26], and the differences of our framework.

2. Preliminaries

2.1. Notations

The parameter $d \in \{1, 2\}$ denotes the horizontal space dimension, $\mathbf{x} \in \mathbb{D}^d$, where $\mathbb{D} = \mathbb{R}$ or $\mathbb{D} = \mathbb{T}$. If $d = 2$, then we denote $\mathbf{x} = (x, y)$. We sometimes assume for simplicity that $d = 2$, the setting $d = 1$ being recovered after straightforward simplifications. Id_d is the $d \times d$ identity matrix while $0_{d_1,d_2}$ is the $d_1 \times d_2$ null matrix.

The notation $a \lesssim b$ means that $a \leqslant C_0\, b$, where C_0 is a nonnegative constant whose exact expression is of no importance. We denote by $C(\lambda_1, \lambda_2, \dots)$ a nonnegative constant depending on the parameters λ_1, λ_2,... and whose dependence on the λ_j is always assumed to be nondecreasing. $\lfloor \cdot \rfloor$ denotes the floor function.

We use standard notations for functional spaces. $L^2(\mathbb{D}^d)$ is the standard Hilbert space of square-integrable functions, associated with the inner-product $\big(f_1, f_2\big)_{L^2} = \int_{\mathbb{D}^d} f_1(\boldsymbol{x}) f_2(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ and the norm $\big|f\big|_{L^2} = \big(\int_{\mathbb{D}^d} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}\big)^{1/2}$. The space $L^\infty(\mathbb{D}^d)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm $\big|f\big|_{L^\infty} = \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{D}^d} |f(\boldsymbol{x})| < \infty$. We endow the space $W^{1,\infty}(\mathbb{D}^d) = \{f,\ \text{s.t.}\ f \in L^\infty(\mathbb{D}^d),\ \nabla f \in L^\infty(\mathbb{D}^d)^d\}$ with its canonical norm. For any real constant $s \in \mathbb{R}$, $H^s(\mathbb{D}^d)$ denotes the Sobolev space of all tempered distributions f with finite norm $\big|f\big|_{H^s} = \big|(1 - \Delta)^{s/2} f\big|_{L^2}$. For $j \in \mathbb{N}$, I a real interval and X a normed space, $\mathcal{C}^j(I; X)$ denotes the space of X-valued continuous functions on I with continuous derivatives up to the order j. All these norms extend to vector-valued functions by using the Euclidean norm

Additionally, we introduce non-standard 'norms', denoted with double-bars: $\big\| \cdot \big\|_{s,m,\tilde\lambda}$, $\big\| \cdot \big\|_{s,m,\tilde\lambda,(1)}$, $\big\| \cdot \big\|_{s,m,\tilde\lambda,(2)}$ defined in (3.1)–(3.3) and $\big\| \cdot \big\|_{s,j}$ defined in (4.1).

When $\mathbf{k} = (k_x, k_y) \in \mathbb{N}^d$ is a multi-index, $|\mathbf{k}| = k_x + k_y$ and $\partial^{\mathbf{k}} = \partial_x^{k_x} \partial_y^{k_y}$ when $d = 2$ (otherwise $d = 1$, $|\mathbf{k}| = k$ and $\partial^{\mathbf{k}} = \partial_x^k$). For X, Y two closed linear operators (typically of differentiation and pointwise multiplication), we denote $[X, Y] \stackrel{\text{def}}{=} XY - YX$ the commutator whose domain is clear from the context.

2.2. Technical tools

We use mostly without reference the standard continuous Sobolev embedding $H^s(\mathbb{D}^d) \subset L^\infty(\mathbb{D}^d)$ for $s > d/2$ with

$$\big|f\big|_{L^\infty} \lesssim \big|f\big|_{H^s}.$$

The following product estimate is proved for instance in [5, theorem C.10].

Lemma 2.1. *Let $f \in H^{s_1}(\mathbb{D}^d)$ and $g \in H^{s_2}(\mathbb{D}^d)$ and $s_1, s_2 \geqslant s_0 \geqslant 0$ such that $s_1 + s_2 > s_0 + d/2$. Then $fg \in H^{s_0}(\mathbb{D}^d)$ and*

$$\big|fg\big|_{H^{s_0}} \lesssim \big|f\big|_{H^{s_1}} \big|g\big|_{H^{s_2}}.$$

In particular, $H^s(\mathbb{D}^d)$ is a Banach algebra as soon as $s > d/2$. We deduce by induction on the number of factors the following multilinear product estimate.

Lemma 2.2. *Let $k \geqslant 2$ and $f_l \in H^{s_l}(\mathbb{D}^d)$ for $l \in \{1, \dots, k\}$, with $s_l \geqslant s_0 \geqslant 0$ and $\sum_{l=1}^k s_l > s_0 + (k-1)d/2$. Then $\prod_l f_l \in H^{s_0}(\mathbb{D}^d)$ and*

$$|\prod_l f_l|_{H^{s_0}} \lesssim \prod_l |f_l|_{H^{s_l}}.$$

We conclude with a technical result concerning the elliptic operator

$$\mathfrak{t}[h] : \varphi \mapsto h^{-3}\varphi - \frac{\mu}{3}\nabla \cdot \left(h^{-1}\nabla\varphi\right).$$

Lemma 2.3. *Let $s > 1 + d/2$ and $\zeta \in H^s(\mathbb{D}^d)$ be such that $1 + \zeta \geqslant h_\star > 0$. Then $\mathfrak{t}[h] : H^1 \to H^{-1}$ is one-to-one and onto. Moreover, one has for any $\psi \in H^k(\mathbb{D}^d)$ with $k \in \mathbb{N}$ such that $k \leqslant s - 1$,*

$$\left|\mathfrak{t}[h]^{-1}\psi\right|_{H^k} + \mu\left|\mathfrak{t}[h]^{-1}\psi\right|_{H^{k+2}} \leqslant C(h_\star^{-1}, \left|\zeta\right|_{H^s})\left|\psi\right|_{H^k}.$$

Proof. The existence and uniqueness of $\varphi \in H^{k+2}(\mathbb{D}^d)$ such that

$$\mathfrak{t}[h]\varphi \stackrel{\text{def}}{=} h^{-3}\varphi - \frac{\mu}{3}\nabla \cdot (h^{-1}\nabla\varphi) = \psi \tag{2.1}$$

follows from standard elliptic theory, and we focus on the estimates. Testing (2.1) against φ yields

$$\left|\varphi\right|_{L^2}^2 + \mu\left|\nabla\varphi\right|_{L^2}^2 \leqslant C(h_\star, \left|h\right|_{L^\infty})\left|\varphi\right|_{L^2}\left|\psi\right|_{L^2}.$$

Using again (2.1), we find

$$\frac{\mu}{3}\left|\Delta\varphi\right|_{L^2} = \left|h\psi - h^{-2}\varphi - \mu h^{-1}\nabla h \cdot \nabla\varphi\right|_{L^2} \leqslant C(h_\star^{-1}, \left|h\right|_{L^\infty}, \mu^{1/2}\left|\nabla h\right|_{L^\infty})\left|\psi\right|_{L^2},$$

and the estimate is proved for $k = 0$. For $1 \leqslant k \leqslant s - 1$, we differentiate (2.1) and find for any $\mathbf{k}$ such that $|\mathbf{k}| = k$,

$$\mathfrak{t}[h]\partial^{\mathbf{k}}\varphi = \partial^{\mathbf{k}}\psi - [\partial^{\mathbf{k}}, h^{-3}]\varphi + \frac{\mu}{3}\nabla \cdot [\partial^{\mathbf{k}}, h^{-1}]\nabla\varphi.$$

Testing against $\partial^{\mathbf{k}}\varphi$, and using lemma 2.2, we have by induction on k

$$\left|\varphi\right|_{H^k} + \mu^{1/2}\left|\varphi\right|_{H^{k+1}} \leqslant C(h_\star^{-1}, \left|\zeta\right|_{H^s})\left|\psi\right|_{H^k}$$

and the result follows by using once again the identity and lemma 2.2. □

2.3. Hyperbolicity of the Favrie–Gavrilyuk system

System (1.4) is a quasilinear system of balance laws. It can be written under the matricial form (in dimension $d = 2$) with $U \stackrel{\text{def}}{=} (\zeta, \mathbf{u}, \eta, w)$:

$$\partial_t U + \mathsf{A}_x(U)\partial_x U + \mathsf{A}_y(U)\partial_y U = F(U).$$

Its principal symbol $\mathsf{L} \stackrel{\text{def}}{=} \mathrm{i}\tau + \mathrm{i}\xi_x\mathsf{A}_x(U) + \mathrm{i}\xi_y\mathsf{A}_y(U)$ is

$$\mathsf{L} = i \begin{pmatrix} \Theta & h\boldsymbol{\xi}^\top & & \\ \alpha\boldsymbol{\xi} & \Theta \mathsf{Id}_d & \beta\boldsymbol{\xi} & \\ & & \Theta & \\ & & & \Theta \end{pmatrix}$$

where $\boldsymbol{\xi} = (\xi_x, \xi_y)^\top$, $\alpha = 1 + \mu\lambda\frac{\eta^2}{3h^3}$, $\beta = \frac{\mu\lambda}{3h}(1 - \frac{2\eta}{h})$ and $\Theta = \tau + u_x\xi_x + u_y\xi_y$. One immediately sees that $\Theta = 0$ solves the characteristic equation, $\det \mathsf{L} = 0$, with multiplicity $d + 1$ and corresponding eigenvectors

$$(0, \xi_y, -\xi_x, 0, 0) \quad ; \quad (\beta, 0, 0, -\alpha, 0) \quad ; \quad (0, 0, 0, 0, 1).$$

The first eigenvector corresponds to the evolution of the vorticity $\omega = curl\mathbf{u}$. The two other components are consistent with the fact that the phase velocity of the linearized Green–Naghdi system vanishes in the high-frequency limit (see [11] for a comparative analysis of the dispersion relation of the Green–Naghdi and Favrie–Gavrilyuk systems). There are two additional values of Θ solving $\det \mathsf{L} = 0$, namely $\Theta = \pm\sqrt{\alpha h}|\boldsymbol{\xi}|$ with corresponding eigenvectors

$$(\mp\sqrt{h}|\boldsymbol{\xi}|, \sqrt{\alpha}\xi_x, \sqrt{\alpha}\xi_y, 0, 0).$$

Hence we see that the system is strongly hyperbolic as soon as one restricts to $U = (\zeta, \mathbf{u}, \eta, w)$ satisfying $\inf(1 + \zeta) \geqslant h_\star > 0$, as its principal symbol is smoothly diagonalizable. As a matter of fact the system is Friedrichs-symmetrizable, since one can exhibit a symmetrizer, S, such that SA_x and SA_y are symmetric:

$$\mathsf{S} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & & \beta & \\ & h\mathsf{Id}_d & & \\ \beta & & \gamma & \\ & & & 1 \end{pmatrix}$$

where γ is taken large enough in order to ensure that S is definite positive as soon as $\inf h \geqslant h_\star > 0$. From this, standard results yield theorem 1.1; see [5].

2.4. Symmetric structure of the Favrie–Gavrilyuk system

We can write the Favrie–Gavrilyuk system with variables $V = (\zeta, \mathbf{u}, \iota, \kappa)$, namely (1.9), in a symmetric matricial form:

$$\mathsf{S}_t(V)\big(\partial_t V + (\mathbf{u}\cdot\nabla)V\big) + \mathsf{S}_x(V)\partial_x V + \mathsf{S}_y(V)\partial_y V = \lambda^{1/2}\mathsf{J}^\mu V + G(V)$$

where

$$\mathsf{S}_t \stackrel{\text{def}}{=} \begin{pmatrix} 3\alpha\beta & & & \\ & 3h\beta\mathsf{Id}_d & & \\ & & h^{-1} & \frac{-\kappa h^2}{(\lambda\mu)^{1/2}} \\ & & \frac{-\kappa h^2}{(\lambda\mu)^{1/2}} & h^3 \end{pmatrix}$$

and

$$\mathsf{S}_x\xi_x + \mathsf{S}_y\xi_y = \begin{pmatrix} 0 & 3h\alpha\beta\boldsymbol{\xi}^\top & & \\ 3h\alpha\beta\boldsymbol{\xi} & 0_{d,d} & \frac{\kappa^2h^2}{(\lambda\mu)^{1/2}}\boldsymbol{\xi} & \\ & \frac{\kappa^2h^2}{(\lambda\mu)^{1/2}}\boldsymbol{\xi}^\top & 0 & \\ & & & 0 \end{pmatrix}$$

with (misusing notation with the preceding section) $\alpha = 1 + \frac{\iota^2}{3h^2}$, $\beta = \frac{1-\frac{\kappa^2h^2}{\lambda\mu}}{1+\frac{2\iota}{(\lambda\mu)^{1/2}h}}$, and

$$G(V) = \begin{pmatrix} 0 \\ \mathbf{0} \\ \mu^{-1/2}h^{-1}\kappa\iota \\ -\mu^{-1/2}h^3\kappa^2 \end{pmatrix}.$$

3. Large time well-posedness

In this section, we provide uniform estimates satisfied by well-prepared strong solutions of the Favrie–Gavrilyuk system (1.4), which yield the large time well-posedness result of theorem 1.3. In the spirit of [26], we define for $m, s \in \mathbb{N}$, $1 \leqslant m \leqslant s$, $\tilde{\lambda} \in (0, +\infty)$ and $V \in \bigcap_{j=0}^{s} \mathcal{C}^j([0,T]; H^{s-j}(\mathbb{D}^d)^{d+3})$

$$\big\|V\big\|_{s,m,\tilde{\lambda}}^2 \stackrel{\text{def}}{=} \sum_{j=0}^{m} \big|\partial_t^j V\big|_{H^{s-j}}^2 + \sum_{j=m+1}^{s} \tilde{\lambda}^{m-j}\big|\partial_t^j V\big|_{H^{s-j}}^2, \tag{3.1}$$

$$\begin{aligned} \big\|V\big\|_{s,m,\tilde{\lambda},(1)}^2 \stackrel{\text{def}}{=} & \sum_{j=0}^{m-1}\sum_{|\mathbf{k}|=0}^{s-j} \big(\mathsf{S}_t(V)\partial_t^j\partial^{\mathbf{k}}V, \partial_t^j\partial^{\mathbf{k}}V\big)_{L^2} \\ & + \sum_{j=m}^{s} \tilde{\lambda}^{m-j}\big(\mathsf{S}_t(V)\partial_t^j V, \partial_t^j V\big)_{L^2}, \end{aligned} \tag{3.2}$$

$$\big\|V\big\|_{s,m,\tilde{\lambda},(2)}^2 \stackrel{\text{def}}{=} \sum_{j=m}^{s-1}\sum_{|\mathbf{k}|=1}^{s-j} \tilde{\lambda}^{m-j}\big|\partial_t^j\partial^{\mathbf{k}}V\big|_{L^2}^2. \tag{3.3}$$

By convention $\big\|V\big\|_{s,m,\tilde{\lambda},(2)} = 0$ if $m = s$. Of course the notation in (3.2) and (3.3) is abusive as the right-hand side does not define a norm. We recall that if $V \in \mathcal{C}^0([0,T]; H^s(\mathbb{D}^d)^{d+3})$ is a strong solutions to (1.9) and $s > 1 + d/2$, then we deduce $V \in \bigcap_{j=0}^{s} \mathcal{C}^j([0,T]; H^{s-j}(\mathbb{D}^d)^{d+3})$, and the above is well-defined and finite.

Before moving on to the statements of the estimates, their proof and their use in the proof of theorem 1.3, we would like to motivate them. As said above, the proof follows very closely the one provided in [26]. We would like to recall this proof, so as to point out where novel difficulties due to our different setting need to be addressed. Recall (see section 2.4) that our system, (1.9), takes the form

$$\mathsf{S}_t(V)\partial_t V + \mathsf{S}_x(V)\partial_x V + \mathsf{S}_y(V)\partial_y V = \lambda^{1/2}\mathsf{J}^\mu V + G(V), \tag{3.4}$$

where $\mathsf{S}_t, \mathsf{S}_x, \mathsf{S}_y$ are smooth functions of V with values into symmetric matrices, J^μ is skew-symmetric and constant-coefficient, and G is a smooth function. The main difference with the

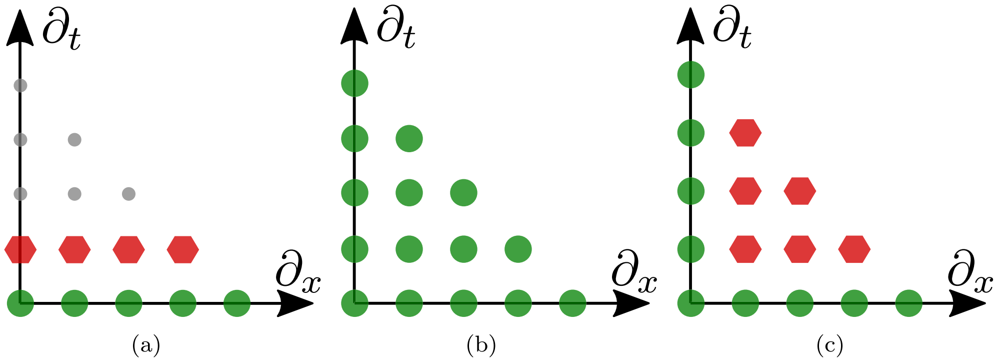

Figure 1. Sketch of the different strategies for *a priori* estimates. (a) Standard. (b) Browning and Kreiss [7]. (c) Schochet [26] Green dots represent space and time derivatives of solutions controlled through energy estimates. Red hexagons represent additional terms whose control is inferred by the system of equations.

framework of [26] is that in our case, J^μ is not homogeneous of order one, but contains an order-zero additional component, and depends on a second parameter μ.

Following the standard strategy for hyperbolic quasilinear systems (which eventually yields theorem 1.1), we first seek a differential inequality for the 'energy' of the system, which after integration in time yields an *a priori* control of the energy for positive times. Thanks to the symmetric structure of the equation, it is immediate to obtain such an estimate, *uniformly with respect to the parameters* $(\lambda, \mu) \in \mathcal{S}_\nu$, by testing the system against V. However this estimate relies on the *a priori* control of the solution itself in L^∞ norm, as well as one derivative with respect to space or time. In other words we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t(V)\,V, V\big)_{L^2} \leqslant C(\big|V\big|_{L^\infty}, \big|\partial_t V\big|_{L^\infty}, \big|\partial_x V\big|_{L^\infty}, \big|\partial_y V\big|_{L^\infty})\big|V\big|_{L^2}^2.$$

In view of obtaining a self-contained energy estimate, the standard strategy consists in differentiating the system with respect to space, and testing against derivatives of the unknown. Thanks to the regularizing properties of commutators, and using the fact that J^μ commutes with space derivatives, one deduces for $s > 1 + d/2$ a uniform differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{|\mathbf{k}|=0}^{s}\big(\mathsf{S}_t(V)\partial^{\mathbf{k}}V, \partial^{\mathbf{k}}V\big)_{L^2}\right) \leqslant C(\big|\partial_t V\big|_{H^{s-1}}, \big|V\big|_{H^s}).$$

For standard (non-singular) quasilinear systems, that is setting $\mathsf{J}^\mu \equiv 0$ in (3.4), the above estimate is sufficient as we have the control

$$\big|\partial_t V\big|_{H^{s-1}} \leqslant C(\big|V\big|_{H^s}) \tag{3.5}$$

stemming from the fact that V satisfies system (3.4), and hence the differential inequality, by Gronwall's lemma and provided that $\mathsf{S}_t(V)$ is positive definite, provides an *a priori* control on $\big|V\big|_{H^s}$. We express the above strategy through the cartoon in figure 1. However, the argument is not useful in our framework as (3.5) is not uniform with respect to $\lambda \gg 1$ due to the contribution from J^μ.

The first strategy that one may have (which is the one developed in [7]) would consist in controlling time derivatives of the unknown through energy estimates as above: differentiating the system with respect to time as well as with space and using that J^μ commutes with space and time derivatives, we obtain (notice we set $m = s$ and hence $\tilde\lambda$ is irrelevant)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|V\big\|^2_{s,s,\tilde\lambda,(1)} \leqslant C(\big\|V\big\|_{s,s,\tilde\lambda}). \tag{3.6}$$

Using that $\big\|V\big\|_{s,s,\tilde\lambda,(1)} \approx \big\|V\big\|_{s,s,\tilde\lambda}$ if $\mathsf{S}_t(V)$ is positive definite, we have indeed a self-contained energy inequality, which can be integrated in time to offer a valuable uniform *a priori* estimate for the solution and derivatives. This is represented in figure 1(b). Notice however that the estimate which is propagated for positive times must of course be satisfied initially: the *a priori* control of $\big\|V\,|_{t=0}\big\|_{s,s,\tilde\lambda}$ is a very strong constraint on the initial data since $m = s$.

The strategy in [26] is more subtle. The first step consists in remarking that, by using energy estimates, we may obtain a uniform energy inequality for time derivatives of the unknowns, of the form (notice we have now $m = 1$ and we set $\tilde\lambda \geq 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|V\big\|^2_{s,1,\tilde\lambda,(1)} \leqslant C(\big\|V\big\|_{s,1,\tilde\lambda}). \tag{3.7}$$

Indeed, it suffices to ensure that only one time derivative of the initial data is uniformly controlled, so that we can take advantage of a gain of a factor $\tilde\lambda^{-1/2}$ as soon as time derivatives are distributed. However, $\big\|V\big\|_{s,1,\tilde\lambda,(1)} \approx \big\|V\big\|_{s,1,\tilde\lambda}$ does not hold, that is we still need to control the contribution of terms involving time *and* space derivatives of the unknowns. To this aim we do not use energy estimates (they fail due to the lack of uniform estimate for $\big|[\partial_t^j\partial^{\mathbf{k}}, \mathsf{S}_t]\partial_t V\big|_{L^2}$ when $j \neq 0$ and $\mathbf{k} \neq \mathbf{0}$) but rather directly control the remaining components with respect to the former:

$$\big\|V\big\|_{s,1,\tilde\lambda,(2)} \leqslant C(\big\|V\big\|_{s,1,\tilde\lambda,(1)}) \tag{3.8}$$

and (under hyperbolicity-type conditions)

$$\big\|V\big\|_{s,1,\tilde\lambda,(1)} + \big\|V\big\|_{s,1,\tilde\lambda,(2)} \approx \big\|V\big\|_{s,1,\tilde\lambda}. \tag{3.9}$$

This is represented in figure 1(c). We cannot infer (3.8) from a simple interpolation uniformly with respect to $\tilde\lambda \gg 1$, but rather will deduce it from system (3.4). This is where the precise properties of J^μ come into play, and this is where our results differ from the ones in [26]. Indeed, when J^μ is a skew-symmetric differential operator, constant-coefficient and homogeneous of order one, we can decompose the (frequency) space as the direct sum of the kernel and the characteristic space associated with non-trival eigenvalues of its symbol. Controlling the projection of V onto the kernel (the 'regular component') is obtained as in (3.5), but applying first the projection to the system onto the kernel, and hence withdrawing the non-uniformly bounded contributions. One controls the other component of V (the 'singular component') in the opposite direction, projecting the system onto the singular subspace and using that the restriction of J^μ to the singular subspace is invertible, and that the inverse is a regularizing operator of order -1 in Sobolev spaces. While the above properties are still true in our setting where J^μ is a non-homogeneous Fourier multiplier, the inverse on the singular subspace is not uniformly bounded with respect to the parameter $\mu \ll 1$ (see lemma 3.4 below). This is easily understood by setting $\mu = 0$, in which case J^μ is an order-zero operator, and hence the inverse cannot be regularizing; and this is the reason why we need to restrict to $\tilde\lambda \in [\nu_\star, \lambda\mu]$ in order to ensure that (3.8) holds uniformly.

The parameter $m \in \{1, \ldots, s\}$ allows to somehow 'interpolate' between the two strategies of [7] and [26], and allows some flexibility on the assumption on the initial data.

Let us now present our results. Proposition 3.1 corresponds to (3.9), while proposition 3.2 corresponds to (3.6) and (3.7) and proposition 3.3 corresponds to (3.8).

In this section, we fix $\lambda, \mu \in (0, +\infty)$ and assume for simplicity that

$$\lambda \geqslant 1 \quad ; \quad \mu \leqslant 1 \quad ; \quad \lambda\mu \geqslant 1.$$

Hence $(\lambda, \mu) \in \mathcal{S}_1$, recalling notation (1.7); the only change in the generally case $(\lambda, \mu) \in \mathcal{S}_\nu$ with $\nu > 0$ is that all constants then depend on the parameter ν.

Proposition 3.1. *Let $s \in \mathbb{N}$ with $s > 1 + d/2$ and $h_\star, h^\star > 0$, $\delta_\star \in (0, 1)$. There exists $C_1 = C(h_\star^{-1}, \delta_\star^{-1}, h^\star)$ such that for any strong solution to (1.9), $V = (\zeta, \mathbf{u}, \iota, \kappa) \in \mathcal{C}^0([0, T]; H^s(\mathbb{D}^d)^{d+3})$ satisfying, uniformly on $[0, T] \times \mathbb{D}^d$, $h \stackrel{\rm def}{=} 1 + \zeta \in [h_\star, h^\star]$,*

$$h|\kappa| \leqslant (1 - \delta_\star)(\lambda\mu)^{1/2} \quad \text{and} \quad 2h^{-1}|\iota| \leqslant (1 - \delta_\star)(\lambda\mu)^{1/2}, \tag{3.10}$$

for any $m \in \mathbb{N}$ such that $1 \leqslant m \leqslant s$ and for any $\tilde{\lambda} \in (0, +\infty)$, one has for any $t \in [0, T]$

$$\frac{1}{C_1} \big\| V \big\|_{s,m,\tilde{\lambda}} \leqslant \big\| V \big\|_{s,m,\tilde{\lambda},(1)} + \big\| V \big\|_{s,m,\tilde{\lambda},(2)} \leqslant C_1 \big\| V \big\|_{s,m,\tilde{\lambda}} \tag{3.11}$$

Proposition 3.2. *Let $s \in \mathbb{N}$ with $s > 1 + d/2$ and $h_\star, M > 0$ and $\delta_\star \in (0, 1)$. There exists $C_2 = C(h_\star^{-1}, \delta_\star^{-1}, M)$ such that for any strong solution to (1.9), $V = (\zeta, \mathbf{u}, \iota, \kappa) \in \mathcal{C}^0([0, T]; H^{s+1}(\mathbb{D}^d)^{d+3})$ satisfying the assumptions of proposition 3.1 uniformly on $[0, T] \times \mathbb{D}^d$ and $\sup_{t\in[0,T]} \big\| V \big\|_{s,m,\tilde{\lambda}} \leqslant M$, for some $m \in \mathbb{N}$ such that $1 \leqslant m \leqslant s$ and for some $\tilde{\lambda} \in [1, +\infty)$, one has for any $t \in [0, T]$*

$$\frac{\rm d}{{\rm d}t} \big\| V \big\|_{s,m,\tilde{\lambda},(1)}^2 \leqslant C_2 \big\| V \big\|_{s,m,\tilde{\lambda}}^3. \tag{3.12}$$

Proposition 3.3. *Let $s \in \mathbb{N}$ with $s > 1 + d/2$ and $h_\star, M, M_{(1)} > 0$. There exists $\nu_\star = C(h_\star^{-1}, M) > 0$ and $C_3 = C(h_\star^{-1}, M_{(1)})$ such that for any strong solution to (1.9), $V = (\zeta, \mathbf{u}, \iota, \kappa) \in \mathcal{C}^0([0, T]; H^{s+1}(\mathbb{D}^d)^{d+3})$ satisfying $h = 1 + \zeta \geqslant h_\star > 0$ uniformly on $[0, T] \times \mathbb{D}^d$, $\sup_{t\in[0,T]} \big\| V \big\|_{s,m,\tilde{\lambda}} \leqslant M$, $\sup_{t\in[0,T]} \big\| V \big\|_{s,m,\tilde{\lambda},(1)} \leqslant M_{(1)}$, for some $m \in \mathbb{N}$ such that $1 \leqslant m \leqslant s$ and some $\tilde{\lambda}$ such that $\lambda\mu \geqslant \tilde{\lambda} \geqslant \nu_\star$, one has for any $t \in [0, T]$*

$$\big\| V \big\|_{s,m,\tilde{\lambda},(2)} \leqslant C_3 \big\| V \big\|_{s,m,\tilde{\lambda},(1)}. \tag{3.13}$$

The proof of proposition 3.1 is an exercise using the explicit formula for S_t given in section 2.4. We postpone the proof of propositions 3.2 and 3.3 to sections 3.2 and 3.3 (respectively), and complete the proof of theorem 1.3 below.

3.1. Proof of theorem 1.3

Let us first assume that the initial data $V_0 \in H^{s+1}(\mathbb{D}^d)$, so that by theorem 1.1—and lemma 2.2 to handle the nonlinear change of variables (1.8)—we have

$$V=(\zeta,\mathbf{u},\iota,\kappa)\in\bigcap_{j=0}^{s}\mathcal{C}^{j+1}([0,T^\star);H^{s-j}(\mathbb{D}^d)^{d+3})$$

and hence all the 'norms' below are well-defined and differentiable on $t\in[0,T^\star)$. We fix $m\in\mathbb{N}$ with $1\leqslant m\leqslant s$, denote $M_{(1)}\stackrel{\rm def}{=}\big\|V\big\|_{s,m,\tilde\lambda,(1)}(0)$ and

$$T^\sharp\stackrel{\rm def}{=}\sup\big\{t\geqslant 0\text{ such that }\big\|V\big\|_{s,m,\tilde\lambda,(1)}(t)\leqslant 2M_{(1)}\text{ and}\\ \forall\mathbf{x}\in\mathbb{D}^d,\quad h_\star/2\leqslant 1+\zeta(t,\mathbf{x})\leqslant 2\big|h(0,\cdot)\big|_{L^\infty}\big\}.$$

By a continuity argument, we have that $T^\sharp>0$. In what follows, we denote M (defined precisely later on) such that

$$\sup_{t\in[0,T^\sharp]}\big\|V\big\|_{s,m,\tilde\lambda}\leqslant M.$$

Propositions 3.1–3.3 yield $\nu_\star=C(2h_\star^{-1},M)>0$ such that for any $\tilde\lambda$ such that $\lambda\mu\geqslant\tilde\lambda\geqslant\nu_\star$, (3.10) holds with $\delta_\star=1/2$ and one has for any $t\in[0,T^\sharp]$:

$$\big\|V\big\|_{s,m,\tilde\lambda}\leqslant C_0\big\|V\big\|_{s,m,\tilde\lambda,(1)},\\ \big\|V\big\|_{s,m,\tilde\lambda,(1)}\leqslant C_1\big\|V\big\|_{s,m,\tilde\lambda}$$

with $C_1=C(2h_\star^{-1},2\big|h(0,\cdot)\big|_{L^\infty})$, $C_3=C(2h_\star^{-1},2M_{(1)})$, $C_0=C_1(1+C_3)$; and

$$\frac{\rm d}{{\rm d}t}\big\|V\big\|_{s,m,\tilde\lambda,(1)}^2\leqslant C_2C_0^3\big\|V\big\|_{s,m,\tilde\lambda,(1)}^3$$

with $C_2=C(2h_\star^{-1},M)$, from which we deduce

$$\big\|V\big\|_{s,m,\tilde\lambda,(1)}\leqslant M_{(1)}\exp(M_{(1)}C_2C_0^3t).$$

At time $t=0$, we have $\big|h\big|_{L^\infty}(0)\lesssim M_0^\star$ and $M_{(1)}\leqslant C_1M_0^\star$ and we may set above $M=2C_0M_{(1)}\leqslant 2C_0C_1M_0^\star=M_0^\star C(h_\star^{-1},M_0^\star)$. We also have by the continuous Sobolev embedding $H^{s-1}\subset L^\infty$ that there exists $c_s>0$ such that for any $t\in[0,T^\sharp]$,

$$1+\zeta(t,x)=1+\zeta(0,x)+\int_0^t\partial_t\zeta(s,x){\rm d}s\in[h_\star-Mc_st,\big|h(0,\cdot)\big|_{L^\infty}+Mc_st].$$

Hence, we deduce by continuity and from the above that $T^\star>T^\sharp\geqslant(M_{(1)}T)^{-1}$ with $T=\sup\{4C_0c_sh_\star^{-1},C_2C_0^3/\ln 2\}=C(h_\star^{-1},M_{(1)})$, which completes the proof when the initial data $V_0\in H^{s+1}(\mathbb{D}^d)$. The general case $V_0\in H^s(\mathbb{D}^d)$ is deduced by a standard regularization and compactness argument; see for instance [26, pp 1631–1632].

The improved result in the setting $m=s$ is proved in the same way, using that propositions 3.1 and 3.2 alone are sufficient to have the necessary estimates and that the initial assumption (3.10) propagates (replacing $\delta_\star$ with $\delta_\star/2$) on the quadratic time scale since $\big|V(t)-V(0)\big|_{L^\infty}\leqslant t\big|\partial_tV\big|_{L^\infty}\lesssim Mc_st$.

3.2. Proof of proposition 3.2

Here and in the following, we denote $V=(\zeta,\mathbf{u},\iota,\kappa)\in\mathcal{C}^0([0,T];H^{s+1}(\mathbb{D}^d))\cap\mathcal{C}^1([0,T];H^s(\mathbb{D}^d))$ a strong solution to (1.9) satisfying $h=1+\zeta\geqslant h_\star>0$. By applying iteratively the equation, one has $\partial_t^j V \in \mathcal{C}^1([0,T];H^{s-j}(\mathbb{D}^d))$ and hence all the terms below are well-defined and continuous with respect to time. Recall (see section 2.4) that (1.9) has the following form when $d=2$:

$$\mathsf{S}_t(V)\partial_t V + \mathsf{S}_x(V)\partial_x V + \mathsf{S}_y(V)\partial_y V = \lambda^{1/2}\mathsf{J}^\mu V + G(V), \tag{3.14}$$

where $\mathsf{S}_t, \mathsf{S}_x, \mathsf{S}_y$ are smooth functions of V with values into symmetric matrices (we simply denote $\mathsf{S}_t, \mathsf{S}_x, \mathsf{S}_y$ for $\mathsf{S}_t(V), \mathsf{S}_x(V), \mathsf{S}_y(V)$ for the sake of conciseness below), J^μ is skew-symmetric, and G is a smooth function. We prove below estimate (3.12) by standard energy method, differentiating (3.14) and testing against derivatives of V.

By testing (3.14) against V and using the symmetry of $\mathsf{S}_t, \mathsf{S}_x, \mathsf{S}_y$ and the skew-symmetry of J^μ, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t V, V\big)_{L^2} = \frac{1}{2}\big([\partial_t, \mathsf{S}_t]V, V\big)_{L^2} + \frac{1}{2}\big([\partial_x, \mathsf{S}_x]V, V\big)_{L^2} + \frac{1}{2}\big([\partial_y, \mathsf{S}_y]V, V\big)_{L^2} + \big(G(V), V\big)_{L^2}.$$

It follows immediately by continuous Sobolev embedding $H^{s-1} \subset L^\infty$ for any $s > 1 + d/2$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t V, V\big)_{L^2} \leqslant C(h_\star^{-1}, \big|V\big|_{H^s})\big(\big|\partial_t V\big|_{H^{s-1}} + \big|V\big|_{H^s} + \mu^{-1/2}\big|\kappa\big|_{L^\infty}\big)\big|V\big|_{L^2}^2.$$

We now control space derivatives of the solution. Given $\mathbf{k} = (k_x, k_y)$ such that $|\mathbf{k}| \leqslant s$, we apply $\partial^{\mathbf{k}} = \partial_x^{k_x}\partial_y^{k_y}$ to (3.14) and test against $\partial^{\mathbf{k}} V$. Because J^μ commutes with space derivatives, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t\partial^{\mathbf{k}} V, \partial^{\mathbf{k}} V\big)_{L^2} = \frac{1}{2}\big(\big([\partial_t, \mathsf{S}_t] + [\partial_x, \mathsf{S}_x] + [\partial_y, \mathsf{S}_y]\big)\partial^{\mathbf{k}} V, \partial^{\mathbf{k}} V\big)_{L^2} + \big([\partial^{\mathbf{k}}, \mathsf{S}_t]\partial_t V + [\partial^{\mathbf{k}}, \mathsf{S}_x]\partial_x V + [\partial^{\mathbf{k}}, \mathsf{S}_y]\partial_y V, \partial^{\mathbf{k}} V\big)_{L^2} + \big(\partial^{\mathbf{k}} G(V), \partial^{\mathbf{k}} V\big)_{L^2}.$$

The first component is estimated as above and we have

$$\big|[\partial_t, \mathsf{S}_t]\partial^{\mathbf{k}} V + [\partial_x, \mathsf{S}_x]\partial^{\mathbf{k}} V + [\partial_y, \mathsf{S}_y]\partial^{\mathbf{k}} V\big|_{L^2} \leqslant C(h_\star^{-1}, \big|V\big|_{H^s})\big(\big|\partial_t V\big|_{H^{s-1}} + \big|V\big|_{H^s}\big)\big|\partial^{\mathbf{k}} V\big|_{L^2}.$$

Using lemma 2.2, we find

$$\big|[\partial^{\mathbf{k}}, \mathsf{S}_t]\partial_t V\big|_{L^2} \leqslant C(h_\star^{-1}, \big|V\big|_{H^s})\big|V\big|_{H^s}\big|\partial_t V\big|_{H^{s-1}},$$

$$\big|[\partial^{\mathbf{k}}, \mathsf{S}_x]\partial_x V\big|_{L^2} + \big|[\partial^{\mathbf{k}}, \mathsf{S}_y]\partial_y V\big|_{L^2} \leqslant C(h_\star^{-1}, \big|V\big|_{H^s})\big|V\big|_{H^s}^2,$$

$$\big|\partial^{\mathbf{k}} G(V)\big|_{L^2} \leqslant C(h_\star^{-1}, \big|V\big|_{H^s})\big(\mu^{-1/2}\big|\iota\big|_{H^{s-1}} + \mu^{-1/2}\big|\kappa\big|_{H^{s-1}}\big)\big|V\big|_{H^s}.$$

Notice that by using the last two equations of (1.9) and since $\lambda\mu \geqslant 1$, we have

$$\mu^{-1/2}\big|\iota\big|_{H^{s-1}} + \mu^{-1/2}\big|\kappa\big|_{H^{s-1}} \leqslant C(h_\star^{-1}, \big|V\big|_{H^s})(\big|\partial_t V\big|_{H^{s-1}} + \big|V\big|_{H^s}).$$

Altogether, and applying Cauchy–Schwarz inequality, we proved

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t\partial^{\mathbf{k}} V, \partial^{\mathbf{k}} V\big)_{L^2} \leqslant C(h_\star^{-1}, \big\|V\big\|_{s,1,\tilde\lambda})\big\|V\big\|_{s,1,\tilde\lambda}^3.$$

The control of the first $m-1$ time derivatives of the solution is identical, using that m time derivatives are uniformly controlled by $\big\|V\big\|_{s,m,\tilde\lambda}$; hence we have

$$\sum_{j=0}^{m-1}\sum_{|\mathbf{k}|=0}^{s-j}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t\partial_t^j\partial^{\mathbf{k}}V,\partial_t^j\partial^{\mathbf{k}}V\big)_{L^2}\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\|V\|_{s,m,\tilde\lambda}^3.$$

Let us now control the time derivatives for $j\geqslant m$. Proceeding as above, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t\partial_t^jV,\partial_t^jV\big)_{L^2}=\frac{1}{2}\big([\partial_t,\mathsf{S}_t]\partial_t^jV+[\partial_x,\mathsf{S}_x]\partial_t^jV+[\partial_y,\mathsf{S}_y]\partial_t^jV,\partial_t^jV\big)_{L^2}$$
$$+\big([\partial_t^j,\mathsf{S}_t]\partial_tV+[\partial_t^j,\mathsf{S}_x]\partial_xV+[\partial_t^j,\mathsf{S}_y]\partial_yV,\partial_t^jV\big)_{L^2}+\big(\partial_t^jG(V),\partial_t^jV\big)_{L^2}.$$

The first terms of the right-hand side are estimated as above:

$$\big|[\partial_t,\mathsf{S}_t]\partial_t^jV+[\partial_x,\mathsf{S}_x]\partial_t^jV+[\partial_y,\mathsf{S}_y]\partial_t^jV\big|_{L^2}$$
$$\leqslant C(h_\star^{-1},|V|_{H^s})\big(|\partial_tV|_{H^{s-1}}+|V|_{H^s}\big)|\partial_t^jV|_{L^2}.$$

The other terms require the use of lemma 2.2 and to pay attention to powers of $\tilde\lambda$. Taking advantage of a gain of a factor $\tilde\lambda^{-m/2}$ as soon as time derivatives are distributed, we find that for any $\tilde\lambda\geqslant 1$, and any $j\geqslant m\geqslant 1$,

$$\tilde\lambda^{\frac{m-j}{2}}\big|[\partial_t^j,\mathsf{S}_t]\partial_tV+[\partial_t^j,\mathsf{S}_x]\partial_xV+[\partial_t^j,\mathsf{S}_y]\partial_yV\big|_{L^2}\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\|V\|_{s,m,\tilde\lambda}^2.$$

Finally, one obtains similarly as above

$$\tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^jG(V)\big|_{L^2}\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\mu^{-1/2}\big(\|\kappa\|_{s-1,m,\tilde\lambda}+\|\iota\|_{s-1,m,\tilde\lambda}\big)\|V\|_{s,m,\tilde\lambda}$$
$$\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\|V\|_{s,m,\tilde\lambda}.$$

Altogether, we proved

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathsf{S}_t\partial_t^jV,\partial_t^jV\big)_{L^2}\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\|V\|_{s,m,\tilde\lambda}^3,$$

for any $j\in\{m,\dots,s\}$. This completes the proof of proposition 3.2.

3.3. Proof of proposition 3.3

This section is dedicated to the proof of proposition 3.3. Contrarily to proposition 3.2, we shall rely strongly on properties of J^μ. Recall our system is of the form (3.14) with

$$\mathsf{J}^\mu=\begin{pmatrix}0&0_{1,d}&&\\0_{d,1}&0_{d,d}&\mu^{1/2}\nabla&\\&\mu^{1/2}\nabla^\top&0&1\\&&-1&0\end{pmatrix}.$$

We introduce $\Pi^{\rm reg}$ and $\Pi^{\rm sing}$ whose symbols are the projections onto the kernel and the eigenspace associated with non-zero eigenvalues of the symbol of J^μ:

$$\Pi^{\rm reg}\stackrel{\rm def}{=}\begin{pmatrix}1&0_{1,d}&0&0\\0_{d,1}&\mathrm{Id}_d+\frac{\mu\nabla\nabla^\top}{1-\mu\Delta}&0_{d,1}&\frac{\mu^{1/2}\nabla}{1-\mu\Delta}\\0&0_{1,d}&0&0\\0&-\frac{\mu^{1/2}\nabla^\top}{1-\mu\Delta}&0&\frac{-\mu\Delta}{1-\mu\Delta}\end{pmatrix},$$

and

$$\Pi^{\rm sing} \stackrel{\rm def}{=} \begin{pmatrix} 0 & 0_{1,d} & 0 & 0 \\ 0_{d,1} & -\frac{\mu\nabla\nabla^\top}{1-\mu\Delta} & 0_{d,1} & -\frac{\mu^{1/2}\nabla}{1-\mu\Delta} \\ 0 & 0_{1,d} & 1 & 0 \\ 0 & \frac{\mu^{1/2}\nabla^\top}{1-\mu\Delta} & 0 & \frac{1}{1-\mu\Delta} \end{pmatrix}.$$

By definition, we have the following properties

$$(\Pi^{\rm sing})^2 = \Pi^{\rm sing}, \qquad (\Pi^{\rm reg})^2 = \Pi^{\rm reg}$$
$$\Pi^{\rm sing} + \Pi^{\rm reg} = \mathsf{Id}_{d+3}, \qquad \Pi^{\rm sing}\Pi^{\rm reg} = 0_{d+3,d+3}$$
$$\Pi^{\rm sing}\,\mathsf{J}^\mu = \mathsf{J}^\mu\,\Pi^{\rm sing} = \mathsf{J}^\mu, \qquad \Pi^{\rm reg}\,\mathsf{J}^\mu = \mathsf{J}^\mu\,\Pi^{\rm reg} = 0_{d+3,d+3}.$$

Thanks to the skew-symmetry of J^μ and the property that the number of non-zero eigenvalues of its symbol does not depend on the (non-zero) frequency (there are always two non-zero eigenvalues and the kernel dimension is $d+1$), we have that $\Pi^{\rm sing}$ and $\Pi^{\rm reg}$ are bounded symmetric operators acting on the Hilbert space $L^2(\mathbb{D}^d)^{d+3}$: for any $U, V \in L^2(\mathbb{D}^d)^{d+3}$,

$$\big(\Pi^{\substack{\rm sing\\ \rm reg}} U, V\big)_{L^2} = \big(U, \Pi^{\substack{\rm sing\\ \rm reg}} V\big)_{L^2} \quad \text{and} \quad \big|V\big|_{L^2}^2 = \big|\Pi^{\rm reg}V\big|_{L^2}^2 + \big|\Pi^{\rm sing}V\big|_{L^2}^2.$$

In the following, we denote $V^{\rm reg} \stackrel{\rm def}{=} \Pi^{\rm reg}V$, $V^{\rm sing} \stackrel{\rm def}{=} \Pi^{\rm sing}V$. Using that $\Pi^{\rm sing}$ and $\Pi^{\rm reg}$ commute with space and time derivatives, we deduce from the above that

$$\big\|V\big\|_{s,m,\tilde\lambda,(2)}^2 \leqslant \sum_{j=m}^{s-1} \tilde\lambda^{m-j}\big|\partial_t^j V^{\rm reg}\big|_{H^{s-j}}^2 + \tilde\lambda^{m-j}\big|\partial_t^j V^{\rm sing}\big|_{H^{s-j}}^2.$$

We provide in the following sections estimates for

$$N^{\rm reg}_{j,k,m,\tilde\lambda} \stackrel{\rm def}{=} \tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j V^{\rm reg}\big|_{H^k} \quad \text{and} \quad N^{\rm sing}_{j,k,m,\tilde\lambda} \stackrel{\rm def}{=} \tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j V^{\rm sing}\big|_{H^k}.$$

The main tool for estimating $N^{\rm sing}_{m,k,j,\tilde\lambda}$ is that, when restricting to the singular subspace, J^μ is a homeomorphism from H^k to H^{k-1}.

Lemma 3.4. *Let $k \in \mathbb{R}$ and $U \in H^{k-1}(\mathbb{D}^d)^{d+3}$ such that $U = \Pi^{\rm sing}U$. Then there exists a unique $V \in H^k(\mathbb{D}^d)^{d+3}$ such that $V = \Pi^{\rm sing}V$ and $U = \mathsf{J}^\mu V$. Moreover, one has $V = \frac{-\mathsf{J}^\mu}{1-\mu\Delta}U$ and in particular there exists a universal constant C_{J^μ} such that*

$$\big|V\big|_{H^k} \leqslant C_{\mathsf{J}^\mu}\mu^{-1/2}\big|U\big|_{H^{k-1}}.$$

3.3.1. Estimate of the singular contribution, $N^{\rm sing}_{j,k,m,\tilde\lambda}$. Differentiating with time the system (3.14), and projecting onto the singular subspace yields the identity for any $j \in \mathbb{N}$:

$$\Pi^{\rm sing}\partial_t^j\left(\mathsf{S}_t(V)\partial_t V + \mathsf{S}_x(V)\partial_x V + \mathsf{S}_y(V)\partial_y V - G(V)\right) = \lambda^{1/2}\Pi^{\rm sing}\mathsf{J}^\mu\Pi^{\rm sing}\partial_t^j V.$$

By distributing the time derivatives, paying attention to powers of $\tilde\lambda$ and using lemma 2.2, we find that for any $0 \leqslant j \leqslant s$ and $k \in \mathbb{N}$ such that $k \leqslant s-j$:

$$\tilde{\lambda}^{\frac{m-j}{2}}\big|\partial_t^j\big(\mathsf{S}_t(V)\partial_t V\big)\big|_{H^{k-1}} \leqslant C(h_\star^{-1},\big|V\big|_{L^\infty})\tilde{\lambda}^{\frac{m-j}{2}}\big|\partial_t^{j+1}V\big|_{H^{k-1}} + C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda}})\big\|V\big\|_{s,m,\tilde{\lambda}}^2.$$

Similarly,

$$\tilde{\lambda}^{\frac{m-j}{2}}\big|\partial_t^j\big(\mathsf{S}_x(V)\partial_x V\big)\big|_{H^{k-1}} \leqslant C(h_\star^{-1},\big|V\big|_{L^\infty})\tilde{\lambda}^{\frac{m-j}{2}}\big|\partial_t^{j}V\big|_{H^{k}} + C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda}})\big\|V\big\|_{s,m,\tilde{\lambda}}^2.$$

Finally,

$$\begin{aligned}\tilde{\lambda}^{\frac{m-j}{2}}\big|\partial_t^j G(V)\big|_{H^{k-1}} &\leqslant C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda}})\mu^{-1/2}\big(\big\|\kappa\big\|_{s-1,m,\tilde{\lambda}}+\big\|\iota\big\|_{s-1,m,\tilde{\lambda}}\big)\big\|V\big\|_{s,m,\tilde{\lambda}}\\ &\leqslant C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda}})\big\|V\big\|_{s,m,\tilde{\lambda}}^2.\end{aligned}$$

Altogether and using lemma 3.4 we deduce

$$N^{\rm sing}_{j,k,m,\tilde{\lambda}} \leqslant C_{\mathsf{J}^\mu}(\lambda\mu)^{-\frac12}C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda},(1)})\big(\tilde{\lambda}^{\frac12}N^{\rm sing}_{j+1,k-1,m,\tilde{\lambda}}+\tilde{\lambda}^{\frac12}N^{\rm reg}_{j+1,k-1,m,\tilde{\lambda}}\big) + C_{\mathsf{J}^\mu}(\lambda\mu)^{-\frac12}C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda}})\big\|V\big\|_{s,m,\tilde{\lambda}}, \tag{3.15}$$

where we recall that $C_{\mathsf{J}^\mu}>0$ is defined in lemma 3.4.

3.3.2. Estimate of the regular component $N^{\rm reg}_{j,k,m,\lambda}$. Now we project system (3.14) onto the regular subspace and apply the differential operator $\partial^{\mathbf{k}}\partial_t^{j-1}$ for $1\leqslant m\leqslant j\leqslant s$ and $\mathbf{k}\in\mathbb{N}^d$ such that $|\mathbf{k}|\leqslant s-j$. Testing against $\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V$ yields

$$\big(\partial^{\mathbf{k}}\partial_t^{j-1}\left(\mathsf{S}_t\partial_t V+\mathsf{S}_x\partial_x V+\mathsf{S}_y\partial_y V-G(V)\right),\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}=0,$$

which we decompose as follows:

$$\begin{aligned}&\big(\mathsf{S}_t\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V,\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}+\big(\mathsf{S}_t\Pi^{\rm sing}\partial^{\mathbf{k}}\partial_t^j V,\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}\\ &\quad+\big(\partial^{\mathbf{k}}\partial_t^{j-1}\big(\mathsf{S}_x\partial_x V+\mathsf{S}_y\partial_y V\big),\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}\\ &\quad+\big(\partial^{\mathbf{k}}[\partial_t^{j-1},\mathsf{S}_t]\partial_t V,\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}\\ &\quad+\big([\partial^{\mathbf{k}},\mathsf{S}_t]\partial_t^j V,\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}\\ &\quad+\big(\partial^{\mathbf{k}}\partial_t^{j-1}G(V),\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2}=0.\end{aligned}$$

Under the assumptions of proposition 3.1, the first contribution gives us the desired control

$$|N^{\rm reg}_{j,k,m,\tilde{\lambda}}|^2\leqslant\tilde{\lambda}^{m-j}C(h_\star^{-1},\delta_\star^{-1},\big|V\big|_{L^\infty})\sum_{|\mathbf{k}|=0}^{k}\big(\mathsf{S}_t\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V,\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^j V\big)_{L^2},$$

and the second contribution is estimated through

$$\tilde{\lambda}^{\frac{m-j}{2}}\big|\mathsf{S}_t\Pi^{\rm sing}\partial^{\mathbf{k}}\partial_t^j V\big|_{L^2}\leqslant C(h_\star^{-1},\big|V\big|_{L^\infty})N^{\rm sing}_{j,k,m,\tilde{\lambda}}.$$

As for the second line, we estimate differently depending on the value of j. If $j\geqslant m+1$, we use the gain of the prefactor $\tilde{\lambda}^{-1/2}$ stemming from the fact that only $j-1$ time derivatives are involved:

$$\tilde{\lambda}^{\frac{m-j}{2}}\big|\partial_t^{j-1}\big(\mathsf{S}_x\partial_x V+\mathsf{S}_y\partial_y V\big)\big|_{H^k}\leqslant\tilde{\lambda}^{\frac{-1}{2}}C(h_\star^{-1},\big\|V\big\|_{s,m,\tilde{\lambda}})\big\|V\big\|_{s,m,\tilde{\lambda}}.$$

When $j=m$, we do not have the gain of the prefactor $\tilde\lambda^{-1/2}$ but less than $m-1$ time derivatives are involved:

$$\tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^{j-1}(\mathsf{S}_x\partial_x V+\mathsf{S}_y\partial_y V)\big|_{H^k}\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda,(1)})\|V\|_{s,m,\tilde\lambda,(1)}.$$

The contribution of the third line is estimated in the same way. As for the contribution of the last line, we deduce from the explicit expression of $\Pi^{\rm reg}$ that

$$\big|\Pi^{\rm reg}\partial^{\mathbf{k}}\partial_t^{j-1}G(V)\big|_{L^2}\leqslant \mu^{\frac12}\big|\partial_t^{j-1}G(V)\big|_{H^{k+1}}$$

hence the contribution of the last line also satisfies the same estimates as above. Finally, the contribution of the fourth line is estimated by

$$\begin{aligned}\tilde\lambda^{\frac{m-j}{2}}\big|[\partial^{\mathbf{k}},\mathsf{S}_t]\partial_t^j V\big|_{L^2}&\leqslant C(h_\star^{-1},|V|_{H^s})\tilde\lambda^{\frac{m-j}{2}}\big|\partial_t^j V\big|_{H^{k-1}}\\&\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda,(1)})\big(N^{\rm reg}_{j,k-1,m,\tilde\lambda}+N^{\rm sing}_{j,k-1,m,\tilde\lambda}\big).\end{aligned}$$

Altogether, by Cauchy–Schwarz inequality, we find for any $m\leqslant j\leqslant s$ and $k\in\mathbb{N}$ such that $k\leqslant s-j$:

$$\begin{aligned}N^{\rm reg}_{j,k,m,\tilde\lambda}\leqslant C(h_\star^{-1},\delta_\star^{-1},\|V\|_{s,m,\tilde\lambda,(1)})\big(N^{\rm sing}_{j,k,m,\tilde\lambda}+N^{\rm reg}_{j,k-1,m,\tilde\lambda}+N^{\rm sing}_{j,k-1,m,\tilde\lambda}+\|V\|_{s,m,\tilde\lambda,(1)}\big)\\+\tilde\lambda^{-1/2}C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\|V\|_{s,m,\tilde\lambda}.\end{aligned}\tag{3.16}$$

3.3.3. Completion. Assuming $\tilde\lambda\leqslant\lambda\mu$ and using that under the assumptions of proposition 3.1, one has

$$\big|N^{\rm reg}_{j,0,m,\tilde\lambda}\big|^2+\big|N^{\rm sing}_{j,0,m,\tilde\lambda}\big|^2=\tilde\lambda^{m-j}\big|\partial_t^j V\big|_{L^2}^2\leqslant C(h_\star^{-1},\delta_\star^{-1},|V|_{L^\infty})\|V\|^2_{s,m,\tilde\lambda,(1)},$$

we immediately deduce from (3.15) and (3.16) that

$$\begin{aligned}\|V\|_{s,m,\tilde\lambda,(2)}&\leqslant C(h_\star^{-1},\delta_\star^{-1},\|V\|_{s,m,\tilde\lambda,(1)})\|V\|_{s,m,\tilde\lambda,(1)}\\&\quad+\tilde\lambda^{-1/2}C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})\|V\|_{s,m,\tilde\lambda}.\end{aligned}$$

It follows that there exists $\nu_\star=C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda})$ such that provided

$$\lambda\mu\geqslant\tilde\lambda\geqslant\nu_\star,$$

the assumptions of proposition 3.1 are satisfied with $\delta_\star=1/2$, and

$$\|V\|_{s,m,\tilde\lambda,(2)}\leqslant C(h_\star^{-1},\|V\|_{s,m,\tilde\lambda,(1)})\|V\|_{s,m,\tilde\lambda,(1)}.$$

Proposition 3.3 is proved.

4. Preparing the initial data

This section is dedicated to the proof of theorem 1.4. As in section 3, we fix $\lambda,\mu\in(0,+\infty)$ and assume for simplicity that

$$\lambda\geqslant 1\quad;\quad\mu\leqslant 1\quad;\quad\lambda\mu\geqslant 1,$$

the general setting being straightforwardly deduced. We shall prove by induction on m that we can set $c^{(j)}$ for $j\in\{1,\cdots,m\}$ such that (1.12) and (1.13) hold. The basic idea consists in iterating the system (1.4) in order to extract explicit expressions for time derivatives in terms of space derivatives, and to iteratively set the corrector terms $c^{(j)}$ so as to cancel out singular (i.e. non-uniformly bounded) terms in these expressions. While it is not difficult to convince oneself, after manipulating the equations and deducing expressions for the first corrector terms, that the induction process may indeed be successfully set up, one quickly realizes that the expressions are very cumbersome. The exact definition for $c^{(j)}$ is provided in (4.7) below, and explicit expressions are provided only for the first-order terms, $c^{(1)}$ and $c^{(2)}$.

We first notice that after differentiating (1.4) with respect to time and using lemma 2.2 (we constantly use this lemma in the following when estimating nonlinear differential operators), one has that any solution $U = (\zeta, \mathbf{u}, \eta, w)$ to (1.4) satisfies[2]

$$\big|\partial_t^{j+1} U\big|_{H^{s-(j+1)}} \leqslant C(h_\star^{-1}, \big\|U\big\|_{s,j})\big(\big\|U\big\|_{s,j} + \lambda\big\|\eta - h\big\|_{s,j}\big). \tag{4.2}$$

Hence we can focus on proving inductively that

$$\lambda\big\|\eta^{(m)} - h^{(m)}\big\|_{s,m}(0) \leqslant M_m$$

with $M_m = C(h_\star^{-1}, M_0)M_0$. Notice that the result for $m = 0$ is trivial and the result for $m = 1$ follows from setting $c^{(1)} = -h_0\nabla\cdot\mathbf{u}_0$, as well as the identity

$$\partial_t(\eta - h) + \mathbf{u}\cdot\nabla(\eta - h) = w + h\nabla\cdot\mathbf{u}. \tag{4.3}$$

Differentiating the above, and applying once again (1.4) on the first-order time derivatives, we find that any solution to (1.4) satisfies

$$\partial_t^2(\eta - h) = \mathfrak{r}[U] + \lambda\mu\mathfrak{s}[U, \eta - h] - \lambda h\mathfrak{t}[h](\eta - h) \tag{4.4}$$

where $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{t}$ are nonlinear differential operators (in space) of order two. The large prefactor that $\lambda\mu$ in front of $\mathfrak{s}$ is compensated by the fact that this operator is quadratic in $\eta - h$, and hence we collect truly singular terms in the operator $\mathfrak{t}$:

$$\mathfrak{t}[h]\varphi = h^{-3}\varphi - \frac{\mu}{3}\nabla\cdot\big(h^{-1}\nabla\varphi\big).$$

For future reference, we also notice that if $U = U_0^{(1)} \stackrel{\text{def}}{=} (\zeta_0, \mathbf{u}_0, h_0, -h_0\nabla\cdot\mathbf{u}_0)$, then $\mathfrak{s}[U_0^{(1)}] = 0$ and

$$\mathfrak{r}[U_0^{(1)}] = h_0\left(\mathbf{u}_0\cdot\nabla(\nabla\cdot\mathbf{u}_0) - (\nabla\cdot\mathbf{u}_0)^2 - \Delta\zeta_0 - \nabla\cdot((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0)\right). \tag{4.5}$$

Rooting from (4.3) and (4.4), we now define

$$\mathfrak{C}_j(U) \stackrel{\text{def}}{=} \partial_t^j\left(\mathfrak{r}[U] + \lambda\mu\mathfrak{s}[U, \eta - h]\right) - \lambda\big[\,\partial_t^j,\, h\mathfrak{t}[h]\,\big](\eta - h)$$

and

$$\mathfrak{S}_m(U) \stackrel{\text{def}}{=} \sum_{k=0}^{\lfloor m/2\rfloor}(-\lambda h\mathfrak{t}[h])^k\mathfrak{C}_{m-2k}(U)$$

[2] Here and below, we denote

$$\big\|U\big\|_{s,m}^2 \stackrel{\text{def}}{=} \sum_{j=0}^{m}\big|\partial_t^j U\big|_{H^{s-j}}^2. \tag{4.1}$$

so that any solution to (1.4) satisfies for any $m \geqslant 2$

$$\partial_t^m(\eta - h) - \mathfrak{S}_{m-2}(U) = \begin{cases} (-\lambda h\mathfrak{t}[h])^{m/2}(\eta - h) & \text{if } m \text{ is even,} \\ (-\lambda h\mathfrak{t}[h])^{(m-1)/2}\partial_t(\eta - h) & \text{if } m \text{ is odd.} \end{cases} \tag{4.6}$$

We deduce the following expression for $c^{(m)}$:

$$(-h_0\mathfrak{t}[h_0])^{\lfloor m/2 \rfloor}c^{(m)} = -\mathfrak{S}_{m-2}(U_0^{(m-1)}) - (-\lambda h_0\mathfrak{t}[h_0])^{\lfloor m/2 \rfloor}s^{(m-2)} \tag{4.7}$$

where $\mathfrak{S}_{m-2}(U_0^{(m-1)})$ is the differential operator of order m obtained when all time derivatives have been replaced by spatial derivatives through (1.4), and

$$s^{(m)} \stackrel{\text{def}}{=} \begin{cases} \sum_{k=1}^{m/2} \lambda^{-k}c^{(2k)} & \text{if } m \text{ is even,} \\ \sum_{k=1}^{(m-1)/2} \lambda^{-k}c^{(2k+1)} - \sum_{k=1}^{(m+1)/2} \lambda^{-k}\mathbf{u}_0 \cdot \nabla c^{(2k)} & \text{if } m \text{ is odd.} \end{cases}$$

Notice $c^{(m)}$ is well-defined by (4.7) and induction on m, using lemma 2.3. We prove below that this choice allows to obtain the desired estimates.

Assuming m is even for simplicity (the case m odd is treated in the same way, with straightforward adjustments), we have by (4.7)

$$(-\lambda h_0\mathfrak{t}[h_0])^{m/2}s^{(m-2)} = -\lambda h_0\mathfrak{t}[h_0]\mathfrak{S}_{m-4}(U_0^{(m-3)}).$$

Now, we have by repeated use of (1.4) and direct product estimates that

$$\begin{aligned} &\big|\lambda h_0\mathfrak{t}[h_0]\big(\mathfrak{S}_{m-4}(U_0^{(m-3)}) - \mathfrak{S}_{m-4}(U_0^{(m-1)})\big)\big|_{H^{s-m}} \\ &\leqslant C'_m\lambda^{1+\frac{m-4}{2}}\Big(\big|U_0^{(m-1)} - U_0^{(m-3)}\big|_{H^{s-m}} + \mu^{\frac{m-2}{2}}\big|U_0^{(m-1)} - U_0^{(m-3)}\big|_{H^{s-2}}\Big) \end{aligned}$$

where $C'_m = C(h_\star^{-1}, \big|U_0^{(m-1)}\big|_{H^s}, \big|U_0^{(m-3)}\big|_{H^s})$. Moreover, we have by definition

$$-\lambda h_0\mathfrak{t}[h_0]\mathfrak{S}_{m-4}(U_0^{(m-1)}) = \mathfrak{S}_{m-2}(U_0^{(m-1)}) - \mathfrak{C}_{m-2}(U_0^{(m-1)})$$

and

$$\big|\mathfrak{C}_{m-2}(U_0^{(m-1)})\big|_{H^{s-m}} \leqslant C_m\Big(\big\|U^{(m-1)}\big\|_{s,m-2} + \lambda\big\|\eta^{(m-1)} - h^{(m-1)}\big\|_{s,m-3}\Big)$$

with $C_m = C(h_\star^{-1}, \big\|U^{(m-1)}\big\|_{s,m-2}, \lambda\big\|\eta^{(m-1)} - h^{(m-1)}\big\|_{s,m-3})$. Combining the above and using the induction hypotheses (1.12) and (1.13), we find

$$\big|\mathfrak{S}_{m-2}(U_0^{(m-1)}) + (-\lambda h_0\mathfrak{t}[h_0])^{m/2}s^{(m-2)}\big|_{H^{s-m}} \leqslant C(h_\star^{-1}, M_0)M_0.$$

It follows that by lemma 2.3 that $c^{(m)}$ is well-defined by (4.7) and satisfies

$$\big|c^{(m)}\big|_{H^{s-m}} + \mu^{m/2}\big|c^{(m)}\big|_{H^s} \leqslant C(h_\star^{-1}, M_0)M_0.$$

Notice that we have in particular, since $\lambda\mu \geqslant 1$, $\big|U_0^{(m)}\big|_{H^s} \leqslant C(h_\star^{-1}, M_0)M_0$. We also observe that for any $j \leqslant m$, one has as above

$$\lambda\big|\mathfrak{S}_{j-2}(U_0^{(m-1)}) - \mathfrak{S}_{j-2}(U_0^{(m)})\big|_{H^{s-j}} \leqslant C(h_\star^{-1}, M_0)M_0.$$

Using this estimate with $j = m$ in (4.7), plugging into (4.6) and using the definition (1.11) shows that with our choice of $c^{(j)}$, one has

$$\lambda\big|\partial_t^m(\eta^{(m)} - h^{(m)})\big|_{H^{s-m}}(0) \leqslant C(h_\star^{-1}, M_0)M_0.$$

The corresponding estimates for time derivatives of lower order are obtained using the estimate directly into (4.6) and using the induction hypothesis. Hence we proved

$$\lambda \big\| \eta^{(m)} - h^{(m)} \big\|_{s,m}(0) \leqslant C(h_\star^{-1}, M_0)M_0,$$

and we deduce from (4.2)

$$\big\| U^{(m)} \big\|_{s,m+1}(0) \leqslant C(h_\star^{-1}, M_0)M_0.$$

This completes the inductive proof of (1.12) and (1.13). We have already displayed $c^{(1)} = -h_0\nabla\cdot\mathbf{u}_0$, and (4.7) with (4.5) yields

$$\begin{aligned}(h_0\mathfrak{t}[h_0])c^{(2)} &= \mathfrak{C}_0(U_0^{(m-1)})\\ &= h_0\left(\mathbf{u}_0\cdot\nabla(\nabla\cdot\mathbf{u}_0) - (\nabla\cdot\mathbf{u}_0)^2 - \Delta\zeta_0 - \nabla\cdot((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0)\right),\end{aligned}$$

from which we deduce the explicit expression for $c^{(2)}$ displayed in theorem 1.4.

5. Conclusion

We have shown the relevance of the Favrie–Gavrilyuk system (1.2) for producing approximate solutions to the Green–Naghdi system (1.1), and ultimately the water-waves system. To this aim, we have exhibited the impact of the shallowness parameter, which may induce undesirable oscillations in space in the shallow-water regime. In order to avoid these oscillations it appears necessary—or at least advisable—to suitably set the initial data for the augmented variables η, w. The following setting is expected to produce good results: set $\lambda \gtrsim gH$ where H is the layer's depth, and given the physical (dimensional) initial data $h\,|_{t=0} = h_0$ and $\mathbf{u}\,|_{t=0} = \mathbf{u}_0$, let

$$w\,|_{t=0} = -h_0\nabla\cdot\mathbf{u}_0 \quad\text{and}\quad \eta\,|_{t=0} = h_0 + \lambda^{-1}c$$

where c is the unique solution to

$$\mathfrak{t}[h_0]c = -g\Delta\zeta_0 + \mathbf{u}_0\cdot\nabla(\nabla\cdot\mathbf{u}_0) - (\nabla\cdot\mathbf{u}_0)^2 - \nabla\cdot((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0)$$

where

$$\mathfrak{t}[h_0]c \stackrel{\text{def}}{=} h_0^{-3}c - \frac{1}{3}\nabla\cdot\left(h_0^{-1}\nabla c\right).$$

We emphasize the fact that w and η are additional variables of the Favrie–Gavrilyuk system, and hence the above choice does not represent any physical restriction on the initial data, solely defined by h_0, $\mathbf{u}_0$. The choice above appears as a good compromise between not preparing the initial data at all (the choice $\eta\,|_{t=0} = h_0$ and $w\,|_{t=0} = 0$ which was made in [11] might be responsible for the observed lack of convergence as λ increases) and a fine preparation of the initial data, that is adding additional corrector terms whose formula quickly increase in complexity.

Whether this choice is suitable in practical cases could be verified through numerical experiments. To our knowledge, the Favrie–Gavrilyuk system has been numerically implemented only in the original work [11] with a particular focus on the incidence of the mesh size, and in [17] where perfectly matched layer (PML) boundary conditions are proposed (incidentally, a study on the existence of solitary wave solutions to the Favrie–Gavrilyuk system is also carried out). A real numerical investigation on the convergence towards the corresponding solution to the Green–Naghdi system as $\lambda \to \infty$, and the influence of the shallow-water parameter and the choice of the initial data is still missing. Proposing a well-adapted (asymptotic preserving) numerical scheme would however most certainly require a tailored analysis in particular due to the fact that the linearized system is not uniformly stable as $\lambda \to \infty$, and exceeds the capabilities of the author. A direct comparison of the two systems on standard benchmarks such as the one set up by Beji and Battjes [4] and Dingemans [8] would of course be extremely beneficial.

Notice the benchmarks above involve a non-trivial topography. Another challenge for future studies on the Favrie–Gavrilyuk system would consist in taking into account such variations in space (or time) of the bottom topography. While the corresponding system is easily derived[3], it yields new singular terms, but with variable coefficients. The strategy used in this work is not applicable due to the lack of commutation of these singular terms with respect to differentiation, and another approach is necessary. We refer to [6] for a related problem. Here again, numerical experiments would provide useful information as to the validity of the Favrie–Gavrilyuk system in this context.

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[3] Let us write it down for the sake of completeness. Denoting $b(t,\mathbf{x})$ the prescribed bottom topography and $h(t,\mathbf{x})$ the water depth (so that $(h+b)(t,\mathbf{x})$ is the surface elevation), the Green–Naghdi system reads (see e.g. [16, 19] when $\partial_t b \equiv 0$ and [12, 13] in the general setting)

$$\begin{cases} \partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + g\nabla(h+b) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathcal{P}[h,b,\mathbf{u}] = \mathbf{0}, \end{cases} \tag{5.1}$$

where

$$\mathcal{P}[h,b,\mathbf{u}] \stackrel{\text{def}}{=} \frac{1}{h}\nabla\Big(h^2\big(\frac{\ddot{h}}{3}+\frac{\ddot{b}}{2}\big)\Big) + \Big(\frac{\ddot{h}}{2}+\ddot{b}\Big)\nabla b.$$

The corresponding Favrie–Gavrilyuk system, following the approach in [11], is

$$\begin{cases} \partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + g\nabla(h+b) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathcal{P}^\lambda[\eta,h,b,\mathbf{u}] = \mathbf{0}, \\ \partial_t \eta + \mathbf{u}\cdot\nabla\eta = w - \frac{3}{2}\dot{b}, \\ \partial_t w + \mathbf{u}\cdot\nabla w = -\frac{\lambda}{h^2}\big(\eta-h\big), \end{cases} \tag{5.2}$$

where

$$\mathcal{P}^\lambda[\eta,h,b,\mathbf{u}] \stackrel{\text{def}}{=} -\frac{\lambda}{3h}\nabla\Big(\frac{\eta}{h}\big(\eta-h\big)\Big) - \frac{\lambda}{2h^2}\big(\eta-h\big)\nabla b + \frac{1}{4}\ddot{b}\nabla b.$$

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