A note on the well-posedness of the one-dimensional multilayer shallow water model

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Abstract

In this short note, we prove by elementary means that the one-dimensional multilayer shallow-water (or Saint-Venant) model for density-stratified fluids is well-posed, provided that (i) the density stratification is stable (*i.e.* the denser fluid is deeper); (ii) each layer has non-vanishing depth; (iii) the shear velocities are small enough.

1 Introduction and main results

In this work, we are concerned with the well-posedness of the following *one-dimensional multilayer Saint-Venant system* for $N \geq 1$ layers of homogeneous fluids:

$$\left\{\begin{array}{l} \partial_t \zeta_n \ + \ \displaystyle\sum_{i=n}^{N} \partial_x (h_i u_i) \ = \ 0, \\ \\ \partial_t u_n \ + \ \displaystyle\sum_{i=1}^{n} \frac{g(\rho_i - \rho_{i-1})}{\rho_n} \partial_x \zeta_n \ + \ \frac{1}{2}\partial_x\big(|u_n|^2\big) \ = \ 0 \ . \end{array}\right. \qquad (n = 1, \dots, N) \tag{1}$$

Here, the unknowns $\zeta_n(t,x)$ and $u_n(t,x)$ (with $n = 1, \dots, N$) represent respectively the deformation of the $n^{\rm th}$ interface and the layer-mean horizontal velocity in the $n^{\rm th}$ layer, at time t and horizontal position $x \in \mathbb{R}$ (see Figure 1); thus (1) is a system of $2N$ coupled evolution equations. We denote by $\rho_n > 0$ the mass density of the fluid in the $n^{\rm th}$ layer, whereas g is the gravitational acceleration. By convention, we set $\rho_0 = 0$, and $\zeta_{N+1}(x)$ is the (fixed and given) bottom topography. Finally, $h_i(t,x) \stackrel{\rm def}{=} d_i + \zeta_i(t,x) - \zeta_{i+1}(t,x)$ is the depth of the $i^{\rm th}$ layer.

Such a system can be formally derived as the governing equations for N layers of immiscible, homogeneous, ideal, incompressible fluids under the influence of gravity, making use of the so-called hydrostatic approximation [19, 20, 18, 15, 17]. It can also be rigorously obtained, after a non-dimensionalizing step, as an asymptotic model in the shallow-water limit (*i.e.* the depth of each layer is assumed to be small when compared with the characteristic wavelength of the flow). Such a derivation has been given in [2] (and references therein for earlier works in less general framework) when $N = 1$, and by the author in [12] when $N = 2$; the latter work is easily extended to an arbitrary number of layers.

The multilayer Saint-Venant system with $\rho_1 = \dots = \rho_N$ (and additional terms due to viscosity) has also been introduced by Audusse in [3] in order to numerically compute in an effective way the Navier-Stokes equation. See also the consequential work in [4, 5], where similar aim is pursued through a fairly different strategy. In their setting, the interfaces between each layer are artificial and do not encompass physical significance.

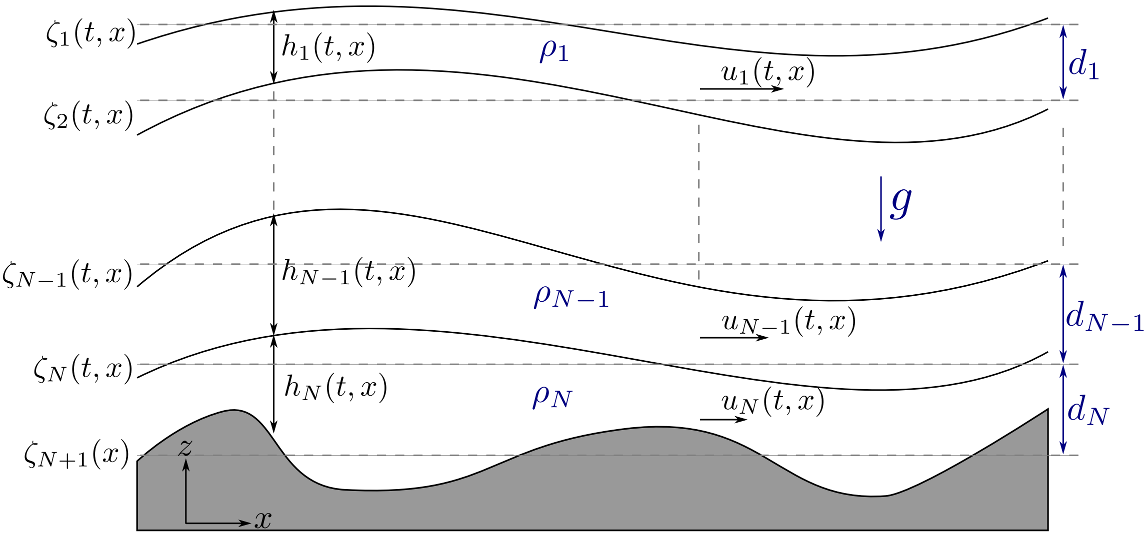

Figure 1: Sketch of the domain and notations

Despite numerous works, the well-posedness of system (1) is, as far as we know, an open question except in very special cases. Of course, our system is quasilinear: sufficiently regular solutions of (1) satisfy $\partial_t \mathcal{V} + A[\mathcal{V}]\partial_x \mathcal{V} = \mathbf{b}[\mathcal{V}]$ where $\mathcal{V} = (\zeta_1, \dots, \zeta_N, u_1, \dots, u_N)^\top$, $\mathbf{b}[\mathcal{V}]$ is a vector component due to bottom topography (which does not play any role in our analysis), and

$$A[\mathcal{V}] \stackrel{\text{def}}{=} \begin{pmatrix} u_1 & u_2 - u_1 & \dots & u_N - u_{N-1} & h_1 & h_2 & \dots & h_N \\ 0 & u_2 & \ddots & \vdots & 0 & h_2 & \dots & h_N \\ \vdots & \ddots & \ddots & u_N - u_{N-1} & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_N & 0 & \dots & 0 & h_N \\ \frac{\rho_1}{\rho_1} & 0 & \dots & 0 & u_1 & 0 & \dots & 0 \\ \frac{\rho_1}{\rho_2} & \frac{\rho_2-\rho_1}{\rho_2} & \ddots & \vdots & 0 & u_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ \frac{\rho_1}{\rho_N} & \frac{\rho_2-\rho_1}{\rho_N} & \dots & \frac{\rho_N-\rho_{N-1}}{\rho_N} & 0 & \dots & 0 & u_N \end{pmatrix}. \tag{2}$$

Thus a natural question is the domain of hyperbolicity of the system. In the case $N = 1$, it is well-known that $A[\mathcal{V}]$ has two distinct, real eigenvalues, provided the depth of the layer is positive; thus the the system is strictly hyperbolic. On the contrary, if $N = 2$ and $\rho_1 = \rho_2$, then the system exhibits complex eigenvalues, except in the case $u_1 = u_2$ [3]. This lack of hyperbolicity is related to Kelvin-Helmholtz instabilities associated with shear flows [16, 6, 9], and indicates that the model is no longer valid and mass exchanges between neighboring layers must be taken into account. On the contrary, when $\rho_1 < \rho_2$, then the hyperbolicity is recovered for sufficiently small shear velocity (namely $u_2 - u_1$); see [23, 10, 21, 13], although the precise domain of hyperbolicity is unknown. To the best of our knowledge, there is no result concerning the case $N \geq 3$, apart from the formal results provided in [14] and a very special case (three layers of equal depth delimited above by a rigid lid) in [11]. Several numerical methods have been proposed that treat the multilayer system even outside its hyperbolic domain [7, 1, 8], although the interpretation and relevance of the computed solutions is unclear in that case.

In this note, we give sufficient conditions, for an arbitrary number of layers, to ensure that the one-dimensional multilayer shallow water system (1) *is strictly hyperbolic, thus well-posed.*

Theorem 1 (Hyperbolicity)**.** *Set $\rho_1, \dots, \rho_N > 0$, $h_1, \dots, h_N > 0$ and $u_1, \dots, u_N \in \mathbb{R}$.*

If $\rho_1 < \rho_2 < \dots < \rho_N$, then there exists $\delta > 0$ such that if

$$\forall i \in \{2, \dots, N\}, \qquad |u_i - u_{i-1}| < \delta \tag{3}$$

is additionally satisfied, then $A[\mathcal{V}]$, given by (2), has $2N$ real, distinct eigenvalues.

On the contrary, if there exists $i \in \{2, \dots, N\}$ such that $\rho_{i-1} > \rho_i$, then for δ sufficiently small, at least two eigenvalues of $A[\mathcal{V}]$ have a non-trivial imaginary part.

Remark 2. *The positive constant δ (or more precisely a lower bound), as well as the eigenvalues, depend continuously on the various parameters at stake; but we are unable to provide an explicit description of this dependency in general. We comment more precisely on that subject in Section 3.*

Corollary 3 (Well-posedness)**.** *Let $s > 3/2$ and $\mathcal{V}^0 \equiv (\zeta_1^0, \dots \zeta_N^0, u_1^0, \dots, u_N^0)^\top \in H^s(\mathbb{R})^{2N}$, and bottom topography $\zeta_{N+1} \in H^{s+1}(\mathbb{R})$. Assume that $0 < \rho_1 < \dots < \rho_N$, and*

$$\forall i \in \{1, \dots, N\}, \qquad \inf_{x \in \mathbb{R}} h_i^0(x) \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}} \big(d_i + \zeta_i^0(x) - \zeta_{i+1}^0(x)\big) > 0.$$

Then there exists constants $\tau, \delta > 0$ such that if

$$\forall i \in \{2, \dots, N\}, \qquad \sup_{x \in \mathbb{R}} |u_i^0(x) - u_{i-1}^0(x)| < \delta,$$

there exists a unique $\mathcal{V} \in C([0,\tau]; H^s)^{2N} \cap C^1([0,\tau]; H^{s-1})^{2N}$ solution to (1), with $\mathcal{V}\,|_{t=0} = \mathcal{V}^0$.

The existence and uniqueness of a strong solution of a strictly hyperbolic system is very classical; see [22] for example. Maybe the only non-standard additional step in the proof of Corollary 3 is to check that the conditions for strict hyperbolicity persist on time interval $[0, \tau]$. This is easily seen when integrating in time (and restricting τ if necessary) the *a priori* energy estimate obtained on $\partial_t \mathcal{V}(t, \cdot) \in H^{s-1}(\mathbb{R})^{2N}$, and making use of Sobolev embeddings.

As usual, one can also deduce from energy estimates that the strong solutions depend continuously on their corresponding initial data, so that system (1) is well-posed in the sense of Hadamard.

2 Proof of Theorem 1

The strategy of our proof relies on perturbative methods. More precisely, we first rewrite $A[\mathcal{V}]$, defined in (2), as $A[\mathcal{V}] \stackrel{\text{def}}{=} A^{(0)} + u_0 \operatorname{Id} + A^{\text{shear}}$, with

$$A^{(0)} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{0} & \begin{matrix} h_1 & h_2 & \dots & h_N \\ 0 & h_2 & \dots & h_N \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & h_N \end{matrix} \\ \begin{matrix} \frac{\rho_1}{\rho_1} & 0 & \dots & 0 \\ \frac{\rho_1}{\rho_2} & \frac{\rho_2 - \rho_1}{\rho_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \frac{\rho_1}{\rho_N} & \frac{\rho_2 - \rho_1}{\rho_N} & \dots & \frac{\rho_N - \rho_{N-1}}{\rho_N} \end{matrix} & \mathbf{0} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{0} & U \\ L & \mathbf{0} \end{pmatrix} \tag{4}$$

and

$$\big\| A^{\text{shear}} \big\| \lesssim \max\{|u_2 - u_1|, \dots, |u_N - u_{N-1}|\}. \tag{5}$$

We then seek conditions for $A^{(0)} + u_0 \operatorname{Id}$ to have $2N$ distinct, real eigenvalues. Equivalently, we give conditions for LU to have N positive distinct eigenvalues, $0 < \lambda_1 < \dots < \lambda_N$.

Lemma 4. *$A^{(0)} + u_0 \operatorname{Id}$ has $2N$ distinct eigenvalues, $\mu^{u_0}_{\pm i}$, if and only if LU has N distinct non-zero eigenvalues, λ_i ($i = 1, \dots, N$). In that case, $\mu^{u_0}_{\pm i} \in \mathbb{R}$ for all $i \in \{1, \dots, N\}$ if and only if $\lambda_i > 0$ for all $i \in \{1, \dots, N\}$ (and one has $\mu^{u_0}_{\pm i} = u_0 \pm \sqrt{\lambda_i}$).*

Proof. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^\top \in \mathbb{C}^{2N}$ with $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^N$. One has

$$\begin{aligned}(A^{(0)} + u_0 \operatorname{Id})\mathbf{x} = \mu \mathbf{x} &\Leftrightarrow A^{(0)}\mathbf{x} = (\mu - u_0)\mathbf{x} \\ &\Leftrightarrow U\mathbf{x}_2 = (\mu - u_0)\mathbf{x}_1 \quad \text{and} \quad L\mathbf{x}_1 = (\mu - u_0)\mathbf{x}_2 \\ &\Rightarrow LU\mathbf{x}_2 = (\mu - u_0)^2 \mathbf{x}_2.\end{aligned}$$

Conversely, assume LU has N distinct non-zero eigenvalues (in particular, L, U are invertible). Let $(\lambda, \mathbf{y})$ satisfying $LU\mathbf{y} = \lambda \mathbf{y}$. Then define μ such that $(\mu - u_0)^2 = \lambda \neq 0$ and $\mathbf{y}' = (\mu - u_0)^{-1} U\mathbf{y}$. Thus $\mathbf{x} = (\mathbf{y}', \mathbf{y})^\top$ satisfies $U\mathbf{y} = (\mu - u_0)\mathbf{y}'$ and $L\mathbf{y}' = (\mu - u_0)^{-1} LU\mathbf{y} = (\mu - u_0)\mathbf{y}$. It follows that $(\mu, \mathbf{x})$ defines an eigenpair of $A^{(0)} + u_0 \operatorname{Id}$.

Lemma 4 is now straightforward. $\square$

Lemma 5. *Assume $\rho_1, \dots, \rho_N > 0$ and $h_1, \dots, h_N > 0$.*

Then at least one of the eigenvalues of LU is zero if and only if there exists $i \in \{1, \dots, N\}$ such that $\rho_i = \rho_{i-1}$ (recall $\rho_0 = 0$ by convention).

Otherwise, LU has N real, distinct eigenvalues: $\lambda_1 < \cdots < \lambda_N$; and $\lambda_1 > 0$ if and only if for all $i \in \{1, \dots, N\}$, one has $\rho_i > \rho_{i-1}$.

Proof. The first point of the statement is straightforward, as $\det(LU) = \prod_{i=1}^N \frac{h_i(\rho_i - \rho_{i-1})}{\rho_i}$. Thereafter, we make use of the assumption $\rho_i \neq \rho_{i-1}$, for any $i \in \{1, \dots, N\}$.

Let us first introduce $\Gamma = \operatorname{diag}(\rho_1, \rho_2, \dots, \rho_N)$, $\Delta = \operatorname{diag}(\rho_1, \rho_2 - \rho_1, \dots, \rho_N - \rho_{N-1})$ and $H = \operatorname{diag}(h_1, \dots, h_N)$ (where $\operatorname{diag}(a_1, \dots, a_N)$ is the N-by-N diagonal matrix whose i^{th} diagonal element is a_i), so that one can decompose

$$LU \ = \ \Gamma^{-1}\ R^\top\ \Delta\ R\ H\ , \quad \text{with} \quad R = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

From this decomposition, one can deduce that the eigenvalues of LU are opposite to the ones of a *symmetric, tri-diagonal, unreduced* matrix (sometimes called *Jacobi matrix*). Indeed, let us introduce

$$T \stackrel{\text{def}}{=} \Gamma^{1/2} H^{-1/2}\ R^{-1}\ \Delta^{-1}\ (R^{-1})^\top\ \Gamma^{1/2} H^{-1/2}.$$

One has $LU\mathbf{x} = \lambda \mathbf{x} \Leftrightarrow T^{-1}(\Gamma^{1/2} H^{1/2}\mathbf{x}) = \lambda^{-1}(\Gamma^{1/2} H^{1/2}\mathbf{x})$, and

$$T = \begin{pmatrix} \frac{\rho_2}{h_1}\frac{1}{\rho_2-\rho_1} & \frac{\sqrt{\rho_1\rho_2}}{\sqrt{h_1h_2}}\frac{-1}{\rho_2-\rho_1} & & & \mathbf{0} \\ \frac{\sqrt{\rho_1\rho_2}}{\sqrt{h_1h_2}}\frac{-1}{\rho_2-\rho_1} & \ddots & \ddots & & \\ & \ddots & \frac{\rho_i}{h_i}\frac{\rho_{i+1}-\rho_{i-1}}{(\rho_{i+1}-\rho_i)(\rho_i-\rho_{i-1})} & \ddots & \\ & & \ddots & \ddots & \frac{\sqrt{\rho_{N-1}\rho_N}}{\sqrt{h_{N-1}h_N}}\frac{-1}{\rho_N-\rho_{N-1}} \\ \mathbf{0} & & & \frac{\sqrt{\rho_{N-1}\rho_N}}{\sqrt{h_{N-1}h_N}}\frac{-1}{\rho_N-\rho_{N-1}} & \frac{\rho_N}{h_N}\frac{1}{\rho_N-\rho_{N-1}} \end{pmatrix}.$$

From the symmetry of T, we deduce classically that all eigenvalues are real. Given our assumptions, T is *unreduced*, *i.e.* its off-diagonal elements are non-zero. In that case, it is well-known that all eigenvalues are distinct. Indeed, if λ^{-1} is an eigenvalue, then $\mathsf{rank}\big(T - \lambda^{-1} \operatorname{Id}\big) = N - 1$, since the submatrix obtained by crossing out the last row and first column has non-zero determinant. Thus λ^{-1} has multiplicity one, and the N eigenvalues are distinct. We have proved

$$\lambda_1 < \cdots < \lambda_N.$$

Let us now conclude with the sign of λ_1 using Sylvester's criterion, *i.e.* looking at the sign of the principal minors of T. It is a simple linear algebra exercise to prove by induction that

$$\forall n \in \{1, \dots, N-1\}, \qquad \det(T_n) = \prod_{i=1}^{n} \frac{\rho_{i+1}}{h_i(\rho_{i+1}-\rho_i)},$$

where T_n is the n-by-n upper-left submatrix (*i.e.* leading principal minor) of T. Finally, one has

$$\det(T) = \prod_{i=1}^{N} \frac{\rho_i}{h_i(\rho_i - \rho_{i-1})} = \frac{1}{h_N}\det(T_{n-1}).$$

The result is now straightforward. □

Theorem 1 is now a direct consequence of the fact that the eigenvalues of the unperturbed matrix $A^{u_0} \stackrel{\text{def}}{=} A^{(0)} + u_0 \,\mathrm{Id}$, namely the solutions of $(\mu^{u_0}_{\pm i} - u_0)^2 = \lambda_i$, are all distinct (using Lemmata 4 and 5), and the continuity of eigenvalues with respect the coefficients of a matrix (and, of course, the control of the smallness of A^{shear} stated in (5)).

Let us give however some additional quantitative details provided by standard perturbation methods [24]. The Bauer–Fike theorem states that, for any eigenvalue, $\mu[A]$, of $A[\mathcal{V}] = A^{u_0} + A^{\text{shear}}$, there exists at least one $j \in \{-N, \dots, -1, 1, \dots, N\}$ such that

$$|\mu[A] - \mu^{u_0}_j| \leq \kappa(H)\big\|A^{\text{shear}}\big\|, \tag{6}$$

where H is the matrix of eigenvectors of A^{u_0} (*i.e.* $H^{-1}A^{u_0}H = \mathrm{diag}(\mu^{u_0}_{\pm i})$), and $\kappa(H) = \big\|H\big\|\big\|H^{-1}\big\|$. Here and below, the only restriction on the matrix norm is that it satisfies $\big\|\mathrm{diag}(a_i)\big\| = \max|a_i|$. Using the explicit expressions given in Lemma 4, the condition number $\kappa(H)$ may be estimated as

$$\kappa(H) \lesssim \sqrt{\frac{\max\rho_i}{\min\rho_i}\frac{\max h_i^3}{\min h_i^3}\frac{\max|\lambda_i|}{\min|\lambda_i|}},$$

where the multiplicative constant depends only on N and the chosen matrix norm.

It is now clear that one can choose $\delta > 0$ sufficiently small such that (5) ensures

$$\kappa(H)\big\|A^{\text{shear}}\big\| < \frac{1}{2}\min_{j,k}|\mu^{u_0}_j - \mu^{u_0}_k|,$$

and therefore each circular disc defined by (6) contains precisely one eigenvalue.

If A^{u_0} has at least one pair of complex conjugate eigenvalues, then it follows that at least two eigenvalues of $A[\mathcal{V}]$ have a non-trivial imaginary part. On the contrary, if all eigenvalues are real (and distinct), then $\Re(\mu_j[A])$ are all distinct and there cannot be any conjugate pairs of eigenvalues of $A[\mathcal{V}]$, *i.e.* its eigenvalues are all real. The proof of Theorem 1 is complete.

3 Conclusion, remarks, perspectives

Using only elementary means, we have been able to give some insight on the seemingly difficult problem of the domain of hyperbolicity of the multi-layer shallow water system. Our result is quite robust, as it works for an arbitrary large number of layers, and the only requirements are the very physical assumptions of a “stable” stratification, non-vanishing depths, small velocity shear.

However, the price to pay is that we do not give any quantitative information. In order to be able, for example, to offer a reasonable lower-bound on δ (which characterize the necessary smallness of the shear velocity in order to ensure well-posedness) or τ (the lifespan of the solution), then we would need to be able to gain some knowledge on the size and the distance between consecutive eigenvalues (the *eigengap*) of the unperturbed operator, A^{u_0}.

This problem is quite difficult, even for symmetric, tri-diagonal matrices. The example of Wilkinson matrices [24] is quite informative in that regard: the tri-diagonal matrices with 1 off-diagonal and $k = -n, \dots, n$ on the diagonal produce impressively close pairs of eigenvalues. For $n = 10$, the two largest eigenvalues are about 10.75 and agree to 14 decimal places!

Another hint as for the sensitivity of our problem with respect to perturbations can be seen as follows. If one replaces the upper-left coefficient of T, $\frac{\rho_2}{h_1}\frac{1}{\rho_2-\rho_1}$, with $\frac{\rho_1}{h_1}\frac{1}{\rho_2-\rho_1}$, then $\nu = 0$ is an eigenvalue of the new matrix, and our whole strategy falls down.

On the other hand, Jacobi matrices are known to enjoy many interesting properties, and one may expect to be able to gain from T some knowledge on the spectral properties of the unperturbed operator A^{u_0} (and consequently of $A[\mathcal{V}]$ for $\mathcal{V}$ sufficiently small).

For example, let us recall that the characteristic polynomial of T_n, the n-by-n upper-left sub-matrix of T —$P_0(X) = 1, P_1(X) = \frac{\rho_2}{h_1}\frac{1}{\rho_2-\rho_1} - X$, *etc.*— are given through a three-term recurrence, and form a *Sturm sequence*: the eigenvalues of T_n interlace strictly with the eigenvalues of T_{n-1} for any $n \in \{2, \dots, N\}$. It follows a convenient way of estimating the location of eigenvalues, as the number of eigenvalues greater than α is then given by the number of agreements of sign between consecutive members of the Sturm sequence $\{P_0(\alpha), P_1(\alpha), \dots, P_N(\alpha)\}$. If not for theoretical purpose, this offers a particularly robust and efficient way of numerically computing the eigenvalues (and consequently eigenvectors), which may be beneficial for the purpose of numerical simulation.

Eigenvectors also enjoy special properties (in addition to the fact that they are of course orthogonal), as they are explicitly given using $P_i(\lambda_j^{-1})$ ($i \in \{1, \dots, N-1\}$) and the off-diagonal elements. Because the off-diagonal elements of T are negative in the case of a stable stratification, one deduces that the eigenvector corresponding to λ_i^{-1} has exactly $i-1$ sign changes. In particular, the eigenvectors of T corresponding to the lowest eigenvalue, *i.e.* λ_1^{-1}, can be chosen to have only positive components, while the other eigenvectors contain components with both signs (and no zero). Physically speaking, the former corresponds to the so-called *barotropic mode*, while the latter correspond to the *baroclinic modes*; see, *e.g.*, [15].

A strategy for extracting more information on our spectral problem can consist in looking at a specific asymptotic limit. A very natural such regime is the so-called *Boussinesq approximation*, in which one assumes that the density contrast between each layer is small (*i.e.* $\frac{\rho_i}{\rho_j} \approx 1$), as is the case in the ocean. In that case, one can easily see that for any $i \geq 2$, $\lambda_i = \mathcal{O}(\rho_1 - \rho_N)$, whereas the λ_1^{-1} is bounded uniformly with respect to $\rho_1 - \rho_N$ (by Courant-Fischer theorem). As a consequence, the *a priori* lifespan of the solution predicted by our naive result vanishes in this limit. It is therefore necessary to characterize further on the spectral projectors associated to the baroclinic and barotropic modes in order to provide extra information concerning the behavior of the flow. Such study has been pursued by the author in the case $N = 2$ in [13].

Another interesting case is the *continuous stratification* setting, namely the case of a density depending continuously with respect to the depth. A natural strategy in our context (both from numerical and theoretical impulse) is to “discretize” the vertical stratification through a great number of layers with constant densities. Our result offers the existence of a solution for an arbitrary large number of layers, provided the stratification is stable and there is no initial shear. The question is then whether one can recover the continuously stratified solution in the limit $N \to \infty$. Of course such property is highly non-trivial, as the density difference between neighboring layers will vanish in this limit, and thus the *a priori* lifespan of the solution as well.

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